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CHARACTERIZATION OF TURING DIFFUSION-DRIVEN INSTABILITY ON EVOLVING DOMAINS

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Abstract. In this paper we establish a general theoretical framework for Turing diffusion-driven instability for reaction-diffusion systems on time-dependent evolving domains. The main result is that Turing diffusion-driven instability for reaction-diffusion systems on evolving domains is characterised by Lyapunov exponents of the evolution family associated with the linearised system (obtained by linearising the original system along a spatially independent solution). This framework allows for the inclusion of the analysis of the long-time behavior of the solutions of reaction-diffusion systems. Applications to two special types of evolving domains are considered: (i) time-dependent domains which evolve to a final limiting fixed domain and (ii) time-dependent domains which are eventually time periodic. Reaction-diffusion systems have been widely proposed as plausible mechanisms for pattern formation in morphogenesis.

1. **Introduction.**

Reaction-diffusion equations (RDEs) have been widely proposed as plausible models of pattern forming processes in developmental biology [27]. On fixed domains, Turing [38] derived the conditions under which a linearised reaction-diffusion system admits a linearly stable spatially homogeneous steady state in the absence of diffusion and yet, becomes unstable under appropriate conditions in the presence of diffusion to yield a spatially varying inhomogeneous steady state. This process is now well-known as diffusively-driven instability and is of particular interest in developmental biological pattern formation as a means of initiating self

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organisation from a virtually homogeneous background. Turing patterns were first observed by Castets *et al.* [2] in a chloride-ionic-malonic-acid (CIMA) reaction and Ouyang and Swinney [32] were the first to observe a Turing diffusion-driven instability from a spatially uniform state to a patterned state. Although the number of actual chemical reactions which produce Turing patterns is small, the idea of diffusion-driven instability, extended by Meinhardt [11] to the paradigm-shifting patterning principle of *short-range activation, long-range inhibition*, has stimulated much biological research in pattern formation. Recently, Shiferaw and Karma [35] proposed a Turing model to describe the interactions between the voltage and calcium in paced cardiac cells. The results of their research provide a striking example of a Turing diffusion-driven instability in a biological context where the morphogens could be identified, as well as a potential link between dynamical instability on subcellular scales and life-threatening cardiac disorders.

On fixed domains, the properties of the autonomous Turing diffusively-driven instability conditions require that the reaction kinetics should be of *activator-inhibitor* form with the inhibitor diffusing faster, typically much faster, than the activator. This gives rise to the standard paradigm of pattern formation via *short-range activation and long-range inhibition.* Most applications of Turing's theory have assumed fixed domains; in the context of developmental biology, this requires the tacit assumption that pattern forming processes occur on a different timescale to that of domain growth. However, recent studies show that domain growth typically dictates the nature of the pattern that evolves as the domain grows leading to a much greater robustness of pattern compared to the array of patterning that can take place on a fixed domain. This is illustrated by Kondo and Asai [13] who predicted mode doubling in pigmentation patterns of the angelfish *Pomacanthus* as it grows. Further examples of studies of RDEs illustrating the role of domain growth can be found in papers by Varea *et al.* [39], Chaplain *et al.* [3], Liaw, *et al.* [14], Painter, *et al.* [28], Crampin *et al.* [6, 5], Oster and Bressloff [31], Madzvamuse, *et al.* [22, 21], Madzvamuse [20] and for a review see Plaza *et al.* [29].

In particular, the latter presented a framework to investigate the role of curvature and growth in pattern formation and selection via the Turing diffusion-driven instability. The corresponding Turing analysis on growing domains was not attempted. Instead, they analysed equations that allow the separation of the geometrical spatial effects from those due to domain growth with the assumption of isotropic linear growth. In all their simulations they observed that the selection of the final pattern was dictated by the interplay of the curvature and domain growth given fixed model parameter values. Transient patterns were shown to be robustly selected due to the effects of either curvature and/or domain growth, in complete agreement with previous results obtained in computational studies by Crampin *et al.* [6, 5] and Madzvamuse *et al.* [22, 20].

As a first step in performing a Turing diffusion-driven instability analysis, we consider the case where RDEs on a growing domain can be transformed into RDEs on a fixed domain, but with time-dependence in the diffusion and dilution terms [6]. These nonautonomous terms however typically invalidate standard linear stability analysis via plane wave decompositions, even with the common simplification that the domain growth is assumed to be isotropic, whereby the domain expands at the same rate in all directions at all times. From a mathematical point of view, stability conditions on fixed domains are typically derived from the calculation of the eigenvalues of a time-independent matrix governing the dynamics of perturbations in the linear regime. Slow growth induces initially small but time-dependent and cumulative changes in such matrices. However, eigenvalues can, in general, be very sensitive to matrix perturbations [12]. By using asymptotic theory, Madzvamuse *et al.* [16] derived and presented a generalisation of the Turing diffusively-driven instability conditions for RDEs on slowly, isotropic growing domains. Only for the case of slowly growing isotropic domains, it was discovered that these conditions are a function of the model parameters in the reaction terms and the growth dynamics.

Despite the above-mentioned studies and the simplifying assumption of slow and isotropic growth of the domain, the analysis presented in [16] is only valid on the slowest timescale found in the model. It is not valid for asymptotically large time, i.e. times much greater than any other timescale in the model. In this paper we aim to establish a general theoretical framework for studying Turing diffusion-driven instability on time-dependent growing domains which allows for the inclusion of long time behaviour of the solutions. We will also consider the special cases that the time-dependent domains has a finite limiting fixed domain or a time-dependent period growing domain. The main focus of this paper is to lay down foundations for future research on deriving Turing diffusion-driven instability conditions for time-dependent domains in terms of the reaction kinetics as well as the model kinetic parameter values. It should however be understood that it is the exception that, as in the case of an *activator-inhibitor* system on a fixed domain, there exists simple algebraic relationships in terms of model parameters for Turing diffusion-driven instability. This feature relies on the fact that the stability analysis leads to eigenvalues problems, where the associated kinetic system reduces to quadratic (possibly cubic) equations. Floquet exponents or more complex systems require serendipitous creativity at a case-by-case level or the use of numerical methods. This is not unique to Turing diffusion-driven instability, but inherited feature from the underlying linear algebra and well-known for systems of ordinary and partial differential equations.

To carry out our studies, we apply the general evolution semigroup or evolution family theory developed in [1], [4] and [34]. Roughly, Turing diffusion-driven instability of RDEs on a growing domain near a spatially homogeneous solution (if it exists) is characterized by Lyapunov exponents of the evolution family associated with the linearized equation at the spatially homogeneous solution, which are analog of eigenvalues of the linearized equation at an equilibrium solution of RDEs on a fixed domain. In fact, if the reaction-diffusion system has a limiting system, this characterization reduces to statements about eigenvalues. Moreover, in another special case, periodically oscillating domains, Lyapunov exponents are identical with the real parts of the classical Floquet or characteristic exponents. Our analysis allows for the inclusion to study limiting systems, i.e. when domain growth saturates to a final fixed domain. This scenario is biologically plausible since most species grow to a finite limiting size as opposed to an infinite domain size. For this case, sufficient diffusion-driven instability conditions are provided in terms of model parameters. The key difference between these results and those obtained on fixed domain is that the diffusion coefficient is scaled by the limiting domain growth profile.

In heart physiology, Turing diffusion-driven instability mediated by voltage and calcium diffusion in paced cardiac cells has been recently studied by Shiferaw and Karma [35]. It turns out that the coupling between the voltage across the cell membrane and the release of the calcium from the intracellular stores is a key ingredient of heart function and this interaction could be modelled by use of reaction-diffusion models. In this paper, we have extended Turing diffusion-driven instability analysis to periodic continuously deforming domains, with period say, T. For this case, we state under what conditions diffusion-driven instability occurs. To our knowledge, this is the first time such a result has been stated and proved. The explicit dependence of these conditions on model parameters is, if not impossible, very hard to find in general.

Hence, our paper is organised as follows: in Section 2 we establish some basic setting for the study of the RDEs on continuously growing domains. First, we transform the partial differential equations to a nonautonomous system of RDEs on a fixed domain and provide some examples which show that the induced systems can have time-independent boundary conditions as well as time-dependent boundary conditions. We then establish some fundamental properties of some linear system associated to the induced system on the fixed domain and some fundamental properties of the induced system. Finally we collect some basic facts about abstract evolution families. Section 3 introduces the definition of Turing diffusion-driven instability on time-dependent domains and explore criteria for such instability. We consider the applications of the general results established in Section 3 to time-varying domains which grow either to a finite limit or change periodically, and state precisely when Turing diffusion-driven instability occurs in Section 4. Finally, we conclude, interpret and discuss the implications of our findings in Section 5.

2. **Basic Setting.**

Consider the following non-dimensionalised system of RDEs on the smooth time-dependent evolving domain $\Omega_t \subset \mathbb{R}^n$ [22, 18, 16],

$$\begin{cases} \frac{\partial u}{\partial t} + \nabla\cdot(\boldsymbol{\beta} u) = \nabla^2 u + \gamma f(u,v), \quad \boldsymbol{x}\in\Omega_t,\ t>s \\ \\ \frac{\partial v}{\partial t} + \nabla\cdot(\boldsymbol{\beta} v) = d_c\nabla^2 v + \gamma g(u,v), \quad \boldsymbol{x}\in\Omega_t,\ t>s \\ \\ \frac{\partial u}{\partial \boldsymbol{\nu}} = \frac{\partial v}{\partial \boldsymbol{\nu}} = 0, \quad \boldsymbol{x}\in\partial\Omega_t,\ t>s \end{cases} \tag{1}$$

where $s \geq 0$, $(u,v) = \big(u(t,\boldsymbol{x}), v(t,\boldsymbol{x})\big) \in \mathbb{R}^2$, $f(u,v)$ and $g(u,v)$ are nonlinear reaction kinetics (for example, $f(u,v) = a - u + u^2 v$, and $g(u,v) = b - u^2 v$ [11, 33]), $\boldsymbol{\beta} = (\beta_1, \beta_2, \cdots, \beta_n)^T$ is the chemical flow velocity, $\boldsymbol{\nu}(\cdot) = (\nu_1(\cdot), \nu_2(\cdot), \cdots, \nu_n(\cdot))^T$ is the unit outer normal vector of $\partial\Omega_t$, and γ represents the scaling positive parameter and d_c is a positive diffusion coefficient.

In this section, we establish some basic setting for the study of the Turing diffusion-driven instability of (1). We first transform (1) to a forward nonautonomous system of reaction-diffusion equations [16] on a fixed domain and provide some examples which show that the induced systems can have time-independent boundary condition as well as time-dependent boundary condition. We then establish some fundamental properties of some linear system associated to the induced system on the fixed domain and some fundamental properties of the induced system. Finally we collect some basic facts about abstract evolution families.

2.1. **Induced system on fixed domain.**

Let $\Omega_0 \subset \mathbb{R}^n$ be a fixed or computational domain. Assume that there is a smooth family of C^3-diffeomorphisms $A_t : \Omega_0 \to \Omega_t$, $\quad \boldsymbol{x} = A_t(y) := \hat{\boldsymbol{x}}(t, \boldsymbol{y})$ which transforms the closure of the fixed domain Ω_0 to the closure of the time-dependent domain Ω_t with $A_t(\partial\Omega_0) = \partial\Omega_t$ and the inverse transformation $\boldsymbol{y} = \hat{\boldsymbol{y}}(t, \boldsymbol{x})$. We refrain from stating specific hypotheses about the domains here, which will be expressed in terms of $\boldsymbol{y}$ later. Let

$$\Big(\hat{u}(t,\boldsymbol{y}), \hat{v}(t,\boldsymbol{y})\Big) = \Big(u(t, \hat{\boldsymbol{x}}(t,\boldsymbol{y})), v(t, \hat{\boldsymbol{x}}(t,\boldsymbol{y}))\Big),$$

and

$$\hat{\boldsymbol{\beta}}(t,\boldsymbol{y}) = \boldsymbol{\beta}\Big(t, \hat{\boldsymbol{x}}(t,\boldsymbol{y})\Big).$$

Then

$$\frac{\partial u}{\partial t} = \frac{\partial \hat{u}}{\partial t} + \sum_{j=1}^{n} \frac{\partial \hat{u}}{\partial y_j}\frac{\partial y_j}{\partial t}, \tag{2}$$

$$\frac{\partial u}{\partial x_i} = \sum_{j=1}^{n} \frac{\partial \hat{u}}{\partial y_j}\frac{\partial y_j}{\partial x_i}, \quad i = 1, 2, \cdots, n, \tag{3}$$

$$\frac{\partial^2 u}{\partial x_i^2} = \sum_{k,j=1}^{n} \frac{\partial^2 \hat{u}}{\partial y_j \partial y_k}\frac{\partial y_j}{\partial x_i}\frac{\partial y_k}{\partial x_i} + \sum_{j=1}^{n} \frac{\partial \hat{u}}{\partial y_j}\frac{\partial^2 y_j}{\partial x_i^2}, \quad i = 1, 2, \cdots, n, \quad \text{and} \tag{4}$$

$$\nabla \cdot \boldsymbol{\beta} = \sum_{i,j=1}^{n} \frac{\partial \hat{\beta}_i}{\partial y_j}\frac{\partial y_j}{\partial x_i}. \tag{5}$$

Hence (1) becomes

$$\begin{cases} \frac{\partial \hat{u}}{\partial t} = \sum_{j,k=1}^{n} a_{j,k}(t,\boldsymbol{y}) \frac{\partial^2 \hat{u}}{\partial y_j \partial y_k} + \sum_{j=1}^{n} \big(b_j(t,\boldsymbol{y}) - c_j(t,\boldsymbol{y})\big) \frac{\partial \hat{u}}{\partial y_j} \\ \qquad + c_0(t,\boldsymbol{y})\hat{u} + \gamma f(\hat{u},\hat{v}), \qquad \boldsymbol{y} \in \Omega_0,\ t > s \\ \\ \frac{\partial \hat{v}}{\partial t} = d_c \sum_{j,k=1}^{n} a_{j,k}(t,\boldsymbol{y}) \frac{\partial^2 \hat{v}}{\partial y_j \partial y_k} + \sum_{j=1}^{n} \big(d_c\, b_j(t,\boldsymbol{y}) - c_j(t,\boldsymbol{y})\big) \frac{\partial \hat{v}}{\partial y_j} \\ \qquad + c_0(t,\boldsymbol{y})\hat{v} + \gamma g(\hat{u},\hat{v}), \qquad \boldsymbol{y} \in \Omega_0,\ t > s \\ \\ \sum_{j=1}^{n} e_j(t,\boldsymbol{y}) \frac{\partial \hat{u}}{\partial y_j} = \sum_{j=1}^{n} e_j(t,\boldsymbol{y}) \frac{\partial \hat{v}}{\partial y_j} = 0, \quad \boldsymbol{y} \in \partial\Omega_0,\ t > s, \end{cases} \tag{6}$$

where $s \geq 0$,

$$a_{j,k}(t,\boldsymbol{y}) = \sum_{i=1}^{n} \frac{\partial y_j}{\partial x_i}\frac{\partial y_k}{\partial x_i}, \quad k = 1, 2, \cdots, n, \tag{7}$$

$$b_j(t,\boldsymbol{y}) = \sum_{i=1}^{n} \frac{\partial^2 y_j}{\partial x_i^2}, \quad j = 1, 2, \cdots, n, \tag{8}$$

$$c_j(t,\boldsymbol{y}) = \frac{\partial y_j}{\partial t} + \sum_{i=1}^{n} \hat{\beta}_i \frac{\partial y_j}{\partial x_i}, \quad j = 1, 2, \cdots, n, \tag{9}$$

$$c_0(t,\boldsymbol{y}) = -\sum_{i,j=1}^{n} \frac{\partial \hat{\beta}_i}{\partial y_j}\frac{\partial y_j}{\partial x_i}, \quad \text{and} \tag{10}$$

$$e_j(t,\boldsymbol{y}) = \sum_{i=1}^{n} \frac{\partial y_j}{\partial x_i} \nu_i(\hat{\boldsymbol{x}}(t,\boldsymbol{y})), \quad j = 1, 2, \cdots, n. \tag{11}$$

System (6) in general depends on t and is defined for $t \geq 0$ only. It is then called *forward nonautonomous*. In the following, we assume that

(H1) *There are* $\alpha_0 > 0$ *and* $M_0 > 0$ *such that*

$$\alpha_0 \|\xi\|^2 \leq \sum_{j,k=1}^{n} a_{j,k}(t, \boldsymbol{y}) \xi_i \xi_j \leq M_0 \|\xi\|^2$$

for any $t \geq 0$, $\boldsymbol{y} \in \Omega_0$, *and* $\xi = (\xi_1, \cdots, \xi_n)^T \in \mathbb{R}^n$.

(H2) $\boldsymbol{e}(t, \boldsymbol{y}) \cdot \hat{\boldsymbol{\nu}}(\boldsymbol{y}) \geq \beta_0 > 0$ *for some* $\beta_0 > 0$ *and all* $t \geq 0$, $\boldsymbol{y} \in \partial\Omega_0$, *where* $\hat{\boldsymbol{\nu}}(\cdot)$ *is the unit outer normal vector of* $\partial\Omega_0$.

(H3) $\Omega_0 \subset \mathbb{R}^n$ is a bounded C^3 domain, and $a_{j,k}$, b_j, c_j, c_0 $(j, k = 1, 2, \cdots, n)$ *are* C^1 *bounded functions on* $[0, \infty) \times \bar{\Omega}_0$ *and* e_j $(j = 1, 2, \cdots, n)$ *are* C^2 *bounded functions on* $[0, \infty) \times \partial\Omega_0$.

2.2. **Examples.**

Example 1. *Time-independent homogeneous Neumann boundary conditions.* Observe that a special case that is widely used is that of uniform isotropic growth, i.e.,

$$\boldsymbol{x} = \rho(t)\boldsymbol{y}. \tag{12}$$

In such a case, we have

$$\begin{aligned}
a_{j,k}(t, \boldsymbol{y}) &= \begin{cases} \frac{1}{\rho(t)^2} & \text{if} \quad j = k, \\ 0 \quad \text{if} & j \neq k, \end{cases} \\
b_j(t, \boldsymbol{y}) &\equiv 0 \quad \text{for} \quad j = 1, 2, \cdots, n, \\
c_j(t, \boldsymbol{y}) &= -\frac{\rho'(t)}{\rho(t)} y_j + \frac{\hat{\beta}_j(t, \boldsymbol{y})}{\rho(t)}, \quad j = 1, 2, \cdots, n, \\
c_0(t, \boldsymbol{y}) &= -\sum_{j=1}^{n} \frac{1}{\rho(t)} \frac{\partial \hat{\beta}_j(t, \boldsymbol{y})}{\partial y_j}, \quad \text{and} \\
e_j(t, \boldsymbol{y}) &= \frac{\nu_j(\boldsymbol{y}\rho(t))}{\rho(t)}, \quad j = 1, 2, \cdots, n.
\end{aligned}$$

Assuming further that the flow velocity is given by $\boldsymbol{\beta} = \frac{\partial \boldsymbol{x}}{\partial t}$ it follows that

$$c_j(t, \boldsymbol{y}) = 0, \quad \text{and} \quad c_0(t, \boldsymbol{y}) = n \frac{\rho'(t)}{\rho(t)}. \tag{13}$$

Example 2. *Time-independent homogeneous Neumann boundary conditions.* Let us assume that domain growth is uniform isotropic and linear given by $\boldsymbol{x} = \rho(t)\boldsymbol{y}$ as before. Assuming further that the flow velocity is different from the mesh (or domain) velocity $\boldsymbol{\beta} \neq \frac{\partial \boldsymbol{x}}{\partial t}$, then we can assume for example a constant chemical flow velocity

$$\hat{\boldsymbol{\beta}}(t, \boldsymbol{y}) = \boldsymbol{\beta}(t, \hat{\boldsymbol{x}}(t, \boldsymbol{y})) = \boldsymbol{\beta} \in \mathbb{R}^n,$$

and as a result convection-reaction-diffusion equations are obtained without advection. For these type of growths, it follows that the boundary conditions are time-independent homogeneous Neumann.

Example 3. *Time-dependent homogeneous Neumann boundary conditions.* Let us assume non-uniform growth of the form

$$x_i = \rho_i(t) y_i, \quad \text{where } \rho_i(0) = 1,\ i = 1, \cdots, n,$$

satisfying $\rho_i(t) \neq \rho_j(t)$ for $i \neq j$. Let the chemical flow velocity be given by [16]

$$\beta_j = \alpha_j + r_j(t) + A_{j,p}(t)x_p + S_{j,p}(t)x_p + B_{j,p,q}(t)x_p x_q$$

where

- α_j represents constant velocity,
- $r_j(t)$ represents body translations (no growth),
- $A_{j,p}(t)$ is antisymmetric and represents a rigid body rotation,
- $S_{j,p}(t)$ is symmetric representing spatially linear growth,
- $B_{j,p,q}(t)$ and all higher order terms represents nonlinear growth.

Clearly it follows that $\boldsymbol{\beta} \neq \frac{\partial \boldsymbol{x}}{\partial t}$ and more importantly, time-dependendent homogeneous Neumann boundary conditions are obtained.

2.3. **Fundamental properties of linear systems.** In this subsection, we establish some fundamental properties of the following forward nonautonomous linear system,

$$\begin{cases} \frac{\partial \hat{u}}{\partial t} = \sum_{j,k=1}^{n} a_{j,k}(t,\boldsymbol{y}) \frac{\partial^2 \hat{u}}{\partial y_j \partial y_k} + \sum_{j=1}^{n} \big(b_j(t,\boldsymbol{y}) - c_j(t,\boldsymbol{y})\big) \frac{\partial \hat{u}}{\partial y_j} \\ \qquad + c_0(t,\boldsymbol{y})\hat{u} + d_{11}(t,\boldsymbol{y})\hat{u} + d_{12}(t,\boldsymbol{y})\hat{v}, \qquad \boldsymbol{y} \in \Omega_0,\ t > s \\ \\ \frac{\partial \hat{v}}{\partial t} = d_c \sum_{j,k=1}^{n} a_{j,k}(t,\boldsymbol{y}) \frac{\partial^2 \hat{v}}{\partial y_j \partial y_k} + \sum_{j=1}^{n} \big(d_c\, b_j(t,\boldsymbol{y}) - c_j(t,\boldsymbol{y})\big) \frac{\partial \hat{v}}{\partial y_j} \\ \qquad + c_0(t,\boldsymbol{y})\hat{v} + d_{21}(t,\boldsymbol{y})\hat{u} + d_{22}(t,\boldsymbol{y})\hat{v}, \qquad \boldsymbol{y} \in \Omega_0,\ t > s \\ \\ \sum_{j=1}^{n} e_j(t,\boldsymbol{y}) \frac{\partial \hat{u}}{\partial y_j} = \sum_{j=1}^{n} e_j(t,\boldsymbol{y}) \frac{\partial \hat{v}}{\partial y_j} = 0, \quad \boldsymbol{y} \in \partial\Omega_0,\ t > s, \end{cases} \tag{14}$$

where $s \geq 0$, $a_{j,k}$, b_j, c_j, c_0, and e_j are as in (6). Note that if $d_{ij} \equiv 0$ for $i, j = 1, 2$, then (14) is the linear part of (6).

Throughout this subsection, assume (H1)-(H3) and d_{11}, d_{12}, d_{21}, d_{22} are C^1 bounded functions on $[0, \infty) \times \bar{\Omega}_0$. For a given Banach space, we denote $\|\cdot\|_X$ as the norm in X. For given Banach spaces X and Y, $\mathcal{L}(X, Y)$ denotes the space of bounded linear operators from X to Y and $\|\cdot\|_{X,Y}$ is the norm in $\mathcal{L}(X, Y)$. We may write $\|\cdot\|_{L^p(\Omega_0)\times L^p(\Omega_0)}$ as $\|\cdot\|_p$ and write $\|\cdot\|_{L^p(\Omega_0)\times L^p(\Omega_0), L^q(\Omega_0)\times L^q(\Omega_0)}$ as $\|\cdot\|_{p,q}$.

The following theorem then follows from the theory developed in [1] (see [1, Theorem 14.5]).

Theorem 2.1. *Let $1 < p < \infty$ and $s \geq 0$.*

(1) (14) *has a $L^p(\Omega_0) \times L^p(\Omega_0)$ solution $(\hat{u}(t,\cdot;s,u_0,v_0), \hat{v}(t,\cdot;s,u_0,v_0))$ for $t > s$ with $(\hat{u}(s,\cdot;s,u_0,v_0), \hat{v}(s,\cdot;s,u_0,v_0)) = (u_0(\cdot), v_0(\cdot)) \in L^p(\Omega_0) \times L^p(\Omega_0)$. Put $U_p(t,s)(u_0,v_0) = (\hat{u}(t,\cdot;s,u_0,v_0), \hat{v}(t,\cdot;s,u_0,v_0)))$ for $t \geq s$ and $(u_0,v_0) \in L^p(\Omega_0) \times L^p(\Omega_0)$.*

(2) There are $M > 0$ and $\omega \in \mathbb{R}$ such that

$$\|U_p(t,s)\|_p \leq M e^{\omega(t-s)} \quad \forall t \geq s.$$

(3) For any $1 < p \leq q < \infty$ and $t \geq s \geq 0$,

$$U_p(t,s)|_{(L^p(\Omega_0)\times L^p(\Omega_0))\cap(L^q(\Omega_0)\times L^q(\Omega_0))} = U_q(t,s)|_{(L^p(\Omega_0)\times L^p(\Omega_0))\cap(L^q(\Omega_0)\times L^q(\Omega_0))}.$$

Theorem 2.2. *If $p > 2n$, then $U_p(t,s)\big(L^p(\Omega_0) \times L^p(\Omega_0)\big) \subset L^q(\Omega_0) \times L^q(\Omega_0)$ for every $t > s$ and $q > p$. Moreover for any given $T > 0$, there are $C > 0$ and $0 < \theta < 1$ such that*

$$\|U_p(t,s)\|_{p,q} \le \frac{C}{(t-s)^\theta} \quad \forall t > s,\ t - s \le T.$$

Proof. First, let X^θ $(0 < \theta < 1)$ be an interpolation space of $L^p(\Omega_0) \times L^p(\Omega_0)$ and $W^{2,p}(\Omega_0) \times W^{2,p}(\Omega_0)$. By the arguments of [1, Theorem 7.1], there is $C > 0$ such that

$$\|U_p(t,s)\|_{L^p(\Omega_0)\times L^p(\Omega_0), X^\theta \times X^\theta} \le \frac{C}{(t-s)^\theta} \quad \forall t > s,\ t - s \le T.$$

Now let $0 < \theta < 1$ be such that $L^q(\Omega_0) \times L^q(\Omega_0) \subset X^\theta$ (such θ exists because $p > 2n$). Then

$$\|U_p(t,s)\|_{p,q} \le \frac{C}{(t-s)^\theta} \quad \forall t > s,\ t - s \le T.$$

□

To indicate the dependence of $U_p(t,s)$ on the coefficients of (14), put

$$a = (a_{j,k}, b_j, c_j, c_0, d_{11}, d_{12}, d_{21}, d_{22}, e_j)$$

and

$$U_p^a(t,s) = U_p(t,s).$$

We make the following assumption.

(H3)$'$ *$a_{j,k}, b_j, c_j, c_0, e_j$ satisfy (H3) and $d_{11}, d_{12}, d_{21}, d_{22}$ are C^1 bounded functions on $[0,\infty) \times \Omega_0$.*

For given
$a = (a_{j,k}, b_j, c_j, c_0, d_{11}, d_{12}, d_{21}, d_{22}, e_j)$ and $\tilde a = (\tilde a_{j,k}, \tilde b_j, \tilde c_j, \tilde c_0, \tilde d_{11}, \tilde d_{12}, \tilde d_{21}, \tilde d_{22}, \tilde e_j)$, we define $d(a, \tilde a)$ by

$$\begin{aligned} d(a,\tilde a) = &\sum_{j,k=1}^n \|a_{j,k} - \tilde a_{j,k}\|_{C^1([0,\infty)\times\Omega_0)} + \\ &+ \sum_{j=1}^n \Big[\|b_j - \tilde b_j\|_{C^1([0,\infty)\times\Omega_0)} + \|c_j - \tilde c_j\|_{C^1([0,\infty)\times\Omega_0)}\Big] + \|c_0 - \tilde c_0\|_{C^1([0,\infty)\times\Omega_0)} \\ &+ \sum_{i,j=1}^2 \|d_{ij} - \tilde d_{ij}\|_{C^1([0,\infty)\times\Omega_0)} + \sum_{j=1}^n \|e_j - \tilde e_j\|_{C^2([0,\infty)\times\partial\Omega_0)}. \end{aligned}$$

In the following, X^θ $(0 < \theta < 1)$ denotes an interpolation space of $L^p(\Omega_0) \times L^p(\Omega_0)$ and $W^{2,p}(\Omega_0) \times W^{2,p}(\Omega_0)$ $(0 < \theta < 1)$.

Theorem 2.3. *For any $1 < p < \infty$, the following hold.*

(1) $[0,\infty) \times [0,\infty) \ni (t,s) \mapsto U_p^a(t+s,s) \in \mathfrak{L}(L^p(\Omega_0) \times L^p(\Omega_0), L^p(\Omega_0) \times L^p(\Omega_0))$ *is strongly continuous. Moreover, for any $0 < \delta_n$ with $\delta_n \to 0$ as $n \to \infty$,*

$$\|U_p^a(\delta_n + t + s, s)(u,v) - U_p^a(t+s,s)(u,v)\|_p \to 0$$

as $n \to \infty$ uniformly in $s \ge 0$, t in bounded subsets of $[0,\infty)$, and (u,v) in bounded subsets of X^θ.

(2) If $d(a^n, a) \to 0$, then $\|U_p^a(t+s,s) - U_p^{a^n}(t+s,s)\|_p \to 0$ as $n \to \infty$ uniformly in $s \in [0,\infty)$ and t in compact subsets of $(0,\infty)$.

Proof. (1) It follows from [34, Theorem 2.2] and [34, Example 2.9].

(2) It follows from [34, Proposition 2.6] and the arguments of [34, Example 2.9]. □

2.4. **Fundamental properties of nonlinear systems.** In this subsection, we establish some fundamental properties of (6). Throughout this subsection, we assume (H1)-(H3). We also assume

(H4) *f and g are C^2 functions. $|f(u,v)|, |g(u,v)| \le M_1(|u|^{p_0} + |v|^{p_0}) + M_2$ and $|\partial_u f(u,v)|$,$|\partial_v f(u,v)|$, $|\partial_u g(u,v)|$, $|\partial_v g(u,v)| \le M_1(|u|^{p_0-1}+|v|^{p_0-1}|)+M_2$ for some $M_1, M_2 > 0$ and $p_0 \ge 1$.*

Choose p, q such that $p > 2n$, $q > pp_0$, and $q > \frac{q(p_0-1)}{q-p}$. Let

$$X = L^q(\Omega_0) \times L^q(\Omega_0). \tag{15}$$

Let $a^0 = (a_{j,k}, b_j, c_j, c_0, 0, 0, 0, 0, e_j)$ and

$$U^0_{p(q)}(t,s) = U^{a^0}_{p(q)}(t,s).$$

Definition 2.4. For given $w_0 \in X$ and $s \ge 0$, $w(t,\cdot) = (u(t,\cdot), v(t,\cdot)) \in C([s, s+T), X)$ is a mild solution of (6) on $[s, s+T)$ with $w(s,\cdot) = w_0(\cdot)$ if

$$w(t,\cdot) = U^0_q(t,s)w_0 + \int_s^t U^0_p(t,\tau)\Big(\gamma f\big(u(\tau,\cdot), v(\tau,\cdot)\big), \gamma g\big(u(\tau,\cdot), v(\tau,\cdot)\big)\Big)^\top d\tau$$

for $t \in [s, s+T)$.

Then we have

Theorem 2.5. *For any $w_0 \in X$ and $s \ge 0$, (6) has a unique (local) mild solution $w(t,\cdot; s, w_0)$ with $w(s,\cdot; s, w_0) = w_0$.*

Proof. First, note that for any $w = (u,v) \in X = L^q(\Omega_0) \times L^q(\Omega_0)$, $(f(u,v), g(u,v)) \in L^p(\Omega_0) \times L^p(\Omega_0)$. Moreover, for any $r > 0$, there is $C(r) > 0$ such that

$$\|(f(u,v), g(u,v)) - (f(\tilde u, \tilde v), g(\tilde u, \tilde v))\|_p \le C(r)\|(u - \tilde u, v - \tilde v)\|_q$$

for (u,v), $(\tilde u, \tilde v) \in X$ with $\|(u,v)\|_q$, $\|(\tilde u, \tilde v)\|_q \le r$.

Next, by Theorem 2.2, for any $T > 0$, there are some $0 < \theta < 1$ and $C > 0$ such that

$$\|U^0_p(t,s)\|_{p,q} \le \frac{C}{(t-s)^\theta}, \quad \forall t \ge s, \quad t - s \le T.$$

The theorem then follows from the results in section 3 of [7] similarly as [7, Corollary 3.7]. □

We remark that in view of more recent results, cf. [8, 9, 15, 41], e.g., Theorem 15.1 of [1] (6) guarantees a unique non-extendable L_q-solution provided that the initial conditions on the boundary satisfy

$$\sum_{j=1}^n e_j(s, \boldsymbol{y})\frac{\partial \hat u}{\partial y_j}(s,x) = \sum_{j=1}^n e_j(s, \boldsymbol{y})\frac{\partial \hat v}{\partial y_j}(s,x) = 0, \qquad \text{for} \quad \boldsymbol{y} \in \partial\Omega_0.$$

Moreover, if, say, the initial values belong additionally to $W^{2,q}(\Omega_0) \times W^{2,q}(\Omega_0)$, then the solution belongs to $W^{1,q}_{\rm loc}(J; X) \cap L^p_{\rm loc}(J; W^{2,q}(\Omega_0) \times W^{2,q}(\Omega_0))$ with J denoting the interval of existence.

2.5. **Evolution family and exponential dichotomy.** In this subsection, we introduce the definitions of evolution family and exponential dichotomy and present some important properties. The reader is referred to [4] for detail.

Let Z be a Banach space and $\mathcal{L}(Z)$ be the set of all the bounded linear operators on Z with the operator norm.

Definition 2.6. A family of operators $\{U(t,s)\}_{t\geq s} \subset \mathcal{L}(Z)$ with $s,t \in \mathbb{R}$ is called an evolution family on Z if

(i) $U(t,s) = U(t,\tau)U(\tau,s)$ and $U(s,s) = I$ for all $t \geq \tau \geq s$; and
(ii) $(t,s) \mapsto U(t,s)$ is strongly continuous for $t \geq s$.

An evolution family $\{U(t,s)\}_{t\geq s}$ on Z is called exponentially bounded if, in addition,

(iii) there exist constants $M > 1$ and $\omega > 0$ such that
$$\|U(t,s)\| \leq Me^{\omega(t-s)} \quad \text{for} \quad t \geq s.$$

Definition 2.7. An exponentially bounded evolution family $\{U(t,s)\}_{t\geq s}$ on Z is called uniformly exponentially stable if its growth rate, defined by
$$\omega(U) := \inf\left\{\omega \,\middle|\, \exists M = M(\omega) \text{ such that } \|U(t,s)\|_{(Z)} \leq Me^{\omega(t-s)} \text{ for all } t \geq s\right\},$$
is negative.

Definition 2.8. An exponentially bounded evolution family $\{U(t,s)\}_{t\geq s}$ on Z is said to have an exponential dichotomy (with constants $M > 0$ and $\beta > 0$) if there exists a projection-valued function $P : \mathbb{R} \to \mathcal{L}(Z)$ such that, for each $z \in Z$, the function $t \mapsto P(t)z$ is continuous and bounded, and, for all $t \geq s$, the following conditions hold:

(i) $P(t)U(t,s) = U(t,s)P(s)$.
(ii) $U_Q(t,s)$ is invertible as an operator from $\text{Im}Q(s)$ to $\text{Im}Q(t)$, where $Q(\cdot) = I - P(\cdot)$ and $U_Q(t,s) = Q(t)U(t,s)Q(s)$.
(iii) $\|U_P(t,s)\| \leq Me^{-\beta(t-s)}$, where $U_P(t,s) = P(t)U(t,s)P(s)$.
(iv) $\|U_Q(t,s)^{-1}\| \leq Me^{-\beta(t-s)}$.

Observe that in Definition 2.8, if $P(\cdot) \equiv I$, then $\{U(t,s)\}_{t\geq s}$ is exponentially stable. In that case, we say $\{U(t,s)\}_{t\geq s}$ admits a *trivial exponential dichotomy*. Otherwise, we say that $\{U(t,s)\}_{t\geq s}$ admits a *nontrivial exponential dichotomy*.

Definition 2.9. Let $\{U(t,s)\}_{t\geq s}$ be an exponentially bounded evolution family on Z. Define an associated evolution semigroup $\{E^t\}_{t\geq 0}$ on $C_0(\mathbb{R};Z)$ as follows:
$$(E^t f)(\theta) = U(\theta,\theta-t)f(\theta-t), \quad t \geq 0, \quad \theta \in \mathbb{R}.$$
$\{E^t\}_{t\geq 0}$ is called the evolution semigroup associated with $\{U(t,s)\}_{t\geq s}$.

Theorem 2.10. *Let $\{U(t,s)\}_{t\geq s}$ be an exponentially bounded evolution family on Z and $\{E^t\}_{t\geq 0}$ be the associated semigroup on $C_0(\mathbb{R};Z)$. Then $\{U(t,s)\}_{t\geq s}$ has exponential dichotomy if and only if $I - E^t$ is invertible for some/all $t > 0$.*

Proof. See [4, Theorems 3.12 and 3.17]. □

Theorem 2.11. *If a strongly continuous, exponentially bounded evolution family $\{U(t,s)\}_{t\geq s}$ on Z has an exponential dichotomy on Z, then, for each $\tau > 0$, there exists an $\epsilon > 0$ such that $\{V(t,s)\}_{t\geq s}$ has an exponential dichotomy whenever*

$\{V(t,s)\}_{t\geq s}$ *is a strongly continuous, exponentially bounded evolution family such that*

$$\sup_{s\in\mathbb{R}} \|V(\tau+s,s)-U(\tau+s,s)\|_{\mathcal{L}(Z)} \leq \epsilon.$$

Proof. See [4, Theorem 5.23]. □

3. Turing Diffusion-Driven Instability.

We are now in a position to introduce in this section the definition of Turing diffusion-driven instability on time-dependent domains and explore criteria for such instability.

We will study the Turing diffusion-driven instability for (1) on time-dependent domain Ω_t by studying the Turing diffusion-driven instability for (6) on the fixed domain Ω_0. Roughly speaking, Turing diffusion-driven instability occurs in (6) means that (6) has a spatially independent solution which is stable in the absence of diffusion, but unstable in the presence of diffusion (see Definition 3.2). Note that in order that (6) has a spatially independent solution, it is necessary that $c_0(t,\boldsymbol{y})$ in (6) is independent of $\boldsymbol{y}$. To study Turing diffusion-driven instability on growing domains, we therefore make the following assumption:

(H5) $c_0(t,\boldsymbol{y}) \equiv c_0(t)$, *i.e.* $c_0(t,\boldsymbol{y})$ *is independent of* $\boldsymbol{y}$.

This assumption is biologically relevant for the case of an isotropic, uniformly growing or evolving domain (see Madzvamuse, *et al.* [16] for specific details). Observe also that equation (5) shows for the case $c_0 \equiv 0$ that this condition is equivalent to the flow field β being incompressible in Ω_t for $t \geq 0$ [17].

In the rest of this subsection, we assume (H1)-(H5) hold. We then have that solutions of the following system of ordinary differential equations

$$\begin{cases} \frac{d\hat{u}}{dt} = c_0(t)\hat{u} + \gamma f(\hat{u},\hat{v}), \quad t>s, \\ \\ \frac{d\hat{v}}{dt} = c_0(t)\hat{v} + \gamma g(\hat{u},\hat{v}), \quad t>s, \end{cases} \tag{16}$$

are also solutions of (6), where $s \geq 0$.

Assume that (16) with $s=0$ has a global bounded solution $(u^*(t),v^*(t))$ (i.e. it exists for all $t>0$ and is bounded on $[0,\infty)$). To study the stability of $(u^*(t),v^*(t))$ in the absence of diffusion as well as in the presence of diffusion, we consider the linearization of (16) at $(u^*(t),v^*(t))$,

$$\begin{cases} \frac{d\hat{u}}{dt} = \Big(c_0(t) + \gamma f_u(u^*(t),v^*(t))\Big)\hat{u} + \gamma f_v(u^*(t),v^*(t))\hat{v}, \quad t>s, \\ \\ \frac{d\hat{v}}{dt} = \gamma g_u(u^*(t),v^*(t))\hat{u} + \Big(c_0(t) + \gamma g_v(u^*(t),v^*(t))\Big)\hat{v}, \quad t>s, \end{cases} \tag{17}$$

and the linearization of (6) at $(u^*(t), v^*(t))$,

$$\begin{cases} \frac{\partial \hat{u}}{\partial t} = \sum_{j,k=1}^n a_{j,k}(t,\boldsymbol{y}) \frac{\partial^2 \hat{u}}{\partial y_j \partial y_k} + \sum_{j=1}^n \Big(b_j(t,\boldsymbol{y}) - c_j(t,\boldsymbol{y}) \Big) \frac{\partial \hat{u}}{\partial y_j} \\ \quad + \Big(c_0(t) + \gamma f_u(u^*(t), v^*(t)) \Big) \hat{u} + \gamma f_v(u^*(t), v^*(t)) \hat{v}, \qquad \boldsymbol{y} \in \Omega_0,\ t > s \\ \\ \frac{\partial \hat{v}}{\partial t} = d_c \sum_{j,k=1}^n a_{j,k}(t,\boldsymbol{y}) \frac{\partial^2 \hat{v}}{\partial y_j \partial y_k} + \sum_{j=1}^n \Big(d_c\, b_j(t,\boldsymbol{y}) - c_j(t,\boldsymbol{y}) \Big) \frac{\partial \hat{v}}{\partial y_j} \\ \quad + \gamma g_u(u^*(t), v^*(t)) \hat{u} + \Big(c_0(t) + \gamma g_v(u^*(t), v^*(t)) \Big) \hat{v}, \qquad \boldsymbol{y} \in \Omega_0,\ t > s \\ \\ \sum_{j=1}^n e_j(t,\boldsymbol{y}) \frac{\partial \hat{u}}{\partial y_j} = \sum_{j=1}^n e_j(t,\boldsymbol{y}) \frac{\partial \hat{v}}{\partial y_j} = 0, \quad \boldsymbol{y} \in \partial\Omega_0,\ t > s, \end{cases} \tag{18}$$

where $s \geq 0$, and f_u, f_v, g_u and g_v represent partial derivatives with respect to u and v, respectively.

It follows from the classical theory for ordinary differential equations that for any $s \geq 0$ and $(u_0, v_0) \in \mathbb{R}^2$, (17) has a unique global solution $(\hat{u}(t; s, u_0, v_0), \hat{v}(t; s, u_0, v_0))$ satisfying the initial condition $(\hat{u}(s; s, u_0, v_0), \hat{v}(s; s, u_0, v_0)) = (u_0, v_0)$. Put

$$\phi(t,s)(u_0, v_0) = \Big(\hat{u}(t; s, u_0, v_0), \hat{v}(t; s, u_0, v_0) \Big). \tag{19}$$

By Theorem 2.1, for any $s \geq 0$ and $(u_0, v_0) \in X$ (X is as in (15)), there is a unique solution $(\hat{u}(t, \boldsymbol{x}; s, u_0, v_0),\ \hat{v}(t, \boldsymbol{x}; s, u_0, v_0))$ of (18) with $(\hat{u}(s, \boldsymbol{x}; s, u_0, v_0),$ $\hat{v}(s, \boldsymbol{x};\ s,\ u_0, v_0)) = (u_0(\boldsymbol{x}), v_0(\boldsymbol{x}))$.

Similarly put

$$\Phi(t,s)(u_0, v_0) = \Big(\hat{u}(t, \boldsymbol{x}; s, u_0, v_0), \hat{v}(t, \boldsymbol{x}; s, u_0, v_0) \Big) \tag{20}$$

for $(u_0, v_0) \in X$.

3.1. **Definitions.** Let X be as in (15), $\Phi(t,s) : X \to X$ be as in (20), and $\phi(t,s) : \mathbb{R}^2 \to \mathbb{R}^2$ be as in (19). In the following, $\|\cdot\|$ denotes either the norm in $\mathbb{R}^2$ or the norm in X. It should be easy to recognize from the context which norm is meant by $\|\cdot\|$.

Definition 3.1. (1) $(u^*(t), v^*(t))$ is called a linearly stable solution of (16) if there are $M > 0$ and $T > 0$ such that for any $s \geq 0$ and $(u_0, v_0) \in \mathbb{R}^2$,

$$\|\phi(t,s)(u_0, v_0)\| \leq M \|(u_0, v_0)\| \quad \text{for} \quad t \geq s + T.$$

$(u^*(t), v^*(t))$ is called a linearly exponentially stable solution of (16) if there are $M > 0$, $T > 0$, and $\delta > 0$ such that for any $s \geq 0$ and $(u_0, v_0) \in \mathbb{R}^2$,

$$\|\phi(t,s)(u_0, v_0)\| \leq M e^{-\delta(t-s)} \|(u_0, v_0)\| \quad \text{for} \quad t \geq s + T.$$

$(u^*(t), v^*(t))$ is a linearly unstable solution of (16) if it is not linearly stable.

(2) $(u^*(t), v^*(t))$ is called a linearly stable solution of (6) if there are $M > 0$ and $T > 0$ such that for any $s \geq 0$ and $(u_0, v_0) \in X$,

$$\|\Phi(t,s)(u_0, v_0)\| \leq M \|(u_0, v_0)\| \quad \text{for} \quad t \geq s + T.$$

$(u^*(t), v^*(t))$ is called a linearly exponentially stable solution of (6) if there are $M > 0$, $T > 0$, and $\delta > 0$ such that for $s \geq 0$ and $(u_0, v_0) \in X$,

$$\|\Phi(t,s)(u_0, v_0)\| \leq M e^{-\delta(t-s)} \|(u_0, v_0)\| \quad \text{for} \quad t \geq s + T.$$

$(u^*(t), v^*(t))$ is a linearly unstable solution of (6) if it is not linearly stable.

Remark 1. In Definition 3.1, $s \geq 0$ can be replaced by $s \geq S$ for some $S > 0$. For example, assume that $(u^*(t), v^*(t))$ is a linearly stable solution of (16) in the sense that there are $M > 0$, $S > 0$, and $T > 0$ such that for any $s \geq S$ and $(u_0, v_0) \in \mathbb{R}^2$,

$$\|\phi(t,s)(u_0,v_0)\| \leq M \quad \text{for} \quad t \geq s + T.$$

Let $M_0 = \sup_{0 \leq s \leq S, s \leq t \leq S} \|\phi(t,s)\|$. Then for any $s \geq 0$ and $t \geq s + S + T$,

$$\|\phi(t,s)(u_0,v_0)\| \begin{cases} \leq M\|(u_0,v_0)\| \quad \text{if} \quad s \geq S, \\ \\ = \|\phi(t,S)\phi(S,s)(u_0,v_0)\| \leq M\|\phi(S,s)(u_0,v_0)\| \leq MM_0\|(u_0,v_0)\| \\ \hfill \text{if} \quad 0 \leq s < S. \end{cases}$$

This implies that for any $s \geq 0$ and $(u_0, v_0) \in \mathbb{R}^2$,

$$\|\phi(t,s)(u_0,v_0)\| \leq \tilde{M}\|(u_0,v_0)\| \quad \text{for} \quad t \geq s + \tilde{T}$$

where $\tilde{M} = (1 + M_0)M$ and $\tilde{T} = S + T$, i.e., $(u^*(t), v^*(t))$ is linearly stable solution of (16) in the sense of Definition 3.1. Other cases can be argued similarly.

Definition 3.2 (Turing Diffusively-Driven Instability on Evolving Domains)**.** One says that Turing *diffusively-driven* instability for (6) near $(u^*(t), v^*(t))$ occurs, if $(u^*(t), v^*(t))$ is linearly exponentially stable in the absence of diffusion and is linearly unstable when diffusion is present.

Results in [24] and [25] (see also [26]) reveal that (forward) Lyapunov exponents of proper forward nonautonomous linear parabolic equations provide an important tool for studying asymptotic dynamics of forward nonautonomous nonlinear parabolic equations. Similarly, (forward) Lyapunov exponents of (17) and (18) provide a useful tool for the study of Turing diffusion-driven instability for (6).

Definition 3.3 (Lyapunov Exponent)**.** $\lambda_+ = \limsup_{t-s\to\infty, s\to\infty} \frac{\ln\|\phi(t,s)\|}{t-s}$ is called the forward top Lyapunov exponent of $\{\phi(t,s)\}_{t\geq s\geq 0}$ or (17).

$\Lambda_+ = \limsup_{t-s\to\infty, s\to\infty} \frac{\ln\|\Phi(t,s)\|}{t-s}$ is called the forward top Lyapunov exponent of $\{\Phi(t,s)\}_{t\geq s\geq 0}$ or (18).

3.2. **Main results.** In this subsection, we explore criteria for Turing diffusion-driven instability. First, we consider the general case. Let λ_+ and Λ_+ be the forward top Lyapunov exponents of (17) and (18), respectively.

Theorem 3.4. *(1) If Turing diffusion-driven instability for* (6) *occurs near* $(u^*(t), v^*(t))$, *then* $\lambda_+ < 0$ *and* $\Lambda_+ \geq 0$.

(2) If $\lambda_+ < 0$ *and* $\Lambda_+ > 0$*, then Turing diffusion-driven instability for* (6) *occurs near* $(u^*(t), v^*(t))$.

Proof. (1) If Turing diffusion-driven instability occurs near $(u^*(t), v^*(t))$, then $(u^*(t), v^*(t))$ is a linearly exponentially stable solution of (16), i.e. there are $M > 0$, $T > 0$, and $\delta > 0$ such that for any $(u_0, v_0) \in \mathbb{R}^2$ and $s \geq 0$,

$$\|\phi(t,s)(u_0,v_0)\| \leq Me^{-\delta(t-s)}\|(u_0,v_0)\| \quad \text{for} \quad t \geq s + T,$$

hence

$$\lambda_+ = \limsup_{t-s\to\infty, s\to\infty} \frac{\ln\|\phi(t,s)\|}{t-s} \leq -\delta < 0.$$

On the other hand, assume that $\limsup_{t-s\to\infty,s\to\infty}\frac{\ln\|\Phi(t,s)\|}{t-s}<0$. Then there are $\delta_0>0$, $s_0>0$, and $T_0>0$ such that

$$\|\Phi(t,s)\|\le e^{-\delta_0(t-s)}\quad\text{for}\quad s\ge s_0,\quad t\ge s+T_0.$$

Let $M_0=1+\sup_{0\le s\le s_0,s\le t\le s_0+T_0}\|\Phi(t,s)\|$, $T=s_0+T_0$, and $M=M_0e^{\delta_0 s_0}$, then

$$\|\Phi(t,s)\|=\|\Phi(t,s_0)\circ\Phi(s_0,s)\|\le M_0e^{-\delta_0(t-s_0)}.$$

for $0\le s\le s_0$ and $t\ge s_0+T_0$, hence

$$\|\Phi(t,s)(u_0,v_0)\|\le Me^{-\delta_0(t-s)}\|(u_0,v_0)\|\quad\text{for}\quad s\ge 0,\quad t\ge s+T,\quad (u_0,v_0)\in X,$$

which implies that $(u^*(t),v^*(t))$ is a linearly exponentially stable solution of (6), a contradiction to Turing diffusion-driven instability occurring.

(2) Suppose that $\lambda_+<0$ and $\Lambda_+>0$. By $\lambda_+<0$ and the arguments in (1), $(u^*(t),v^*(t))$ is a linearly exponentially stable solution of (16).

Assume that $(u^*(t),v^*(t))$ is linearly stable solution of (6). Then there are $M>0$ and $T>0$ such that for any $(u_0,v_0)\in X$ and $s\ge 0$,

$$\|\Phi(t,s)(u_0,v_0)\|\le M\|(u_0,v_0)\|\quad\text{for}\quad t\ge s+T.$$

This implies

$$\Lambda_+=\limsup_{t-s\to\infty,s\to\infty}\frac{\ln\|\Phi(t,s)\|}{t-s}\le 0,$$

which contradicts $\Lambda_+>0$. Therefore, $(u^*(t),v^*(t))$ is a linearly unstable solution of (6) and Turing diffusion-driven instability occurs near $(u^*(t),v^*(t))$. □

Remark 2. Suppose that (6) and (16) are actually time-independent systems and $(u^*(t),v^*(t))\equiv(u^*,v^*)$ is a constant solution of (16), then (u^*,v^*) is a linearly exponentially stable solution of (16) if and only if for any $\lambda\in\sigma(\boldsymbol{J}_F)$, $\mathrm{Re}\lambda<0$, where $\sigma(\boldsymbol{J}_F)$ is the spectrum of $\boldsymbol{J}_F$ and

$$\boldsymbol{J}_F=\begin{pmatrix}c_0+\gamma f_u(u^*,v^*) & \gamma f_v(u^*,v^*)\\ \gamma g_u(u^*,v^*) & c_0+\gamma g_v(u^*,v^*)\end{pmatrix}$$

which is equivalent to $\lambda_+=\limsup_{t-s\to\infty,s\to\infty}\frac{\ln\|\phi(t,s)\|}{t-s}<0$. (u^*,v^*) is a linearly unstable solution of (6) if there is an eigenvalue λ of the following eigenvalue problem with $\mathrm{Re}\lambda>0$,

$$\begin{cases}\sum_{j,k=1}^n a_{j,k}(\boldsymbol{y})\frac{\partial^2\hat u}{\partial y_j\partial y_k}+\sum_{j=1}^n\big(b_j(\boldsymbol{y})-c_j(\boldsymbol{y})\big)\frac{\partial\hat u}{\partial y_j}\\ \qquad +(c_0+\gamma f_u(u^*,v^*))\hat u+\gamma f_v(u^*,v^*)\hat v=\lambda\hat u, & \boldsymbol{y}\in\Omega_0\\ \\ d_c\sum_{j,k=1}^n a_{j,k}(\boldsymbol{y})\frac{\partial^2\hat v}{\partial y_j\partial y_k}+\sum_{j=1}^n\big(d_c\, b_j(\boldsymbol{y})-c_j(\boldsymbol{y})\big)\frac{\partial\hat v}{\partial y_j}\\ \qquad +\gamma g_u(u^*,v^*)\hat u+(c_0+\gamma g_v(u^*,v^*))\hat v=\lambda\hat v, & \boldsymbol{y}\in\Omega_0\\ \\ \sum_{j=1}^n e_j(t,\boldsymbol{y})\frac{\partial\hat u}{\partial y_j}=\sum_{j=1}^n e_j(t,\boldsymbol{y})\frac{\partial\hat v}{\partial y_j}=0, & \boldsymbol{y}\in\partial\Omega_0,\end{cases}\tag{21}$$

which is equivalent to that

$$\Lambda_+=\limsup_{t-s\to\infty,s\to\infty}\frac{\ln\|\Phi(t,s)\|}{t-s}>0.$$

Next, we consider the case that the coefficients of (6) have limits as $t \to \infty$ in the proper sense. Let

$$C^1_{\rm unif}(\mathbb{R},\mathbb{R}) = \Big\{ u : \mathbb{R} \to \mathbb{R} \,\Big|\, u(\cdot), u'(\cdot) \text{ are uniformly continuous and bounded on } \mathbb{R} \Big\}.$$

Suppose that there are $u^\infty(\cdot)$, $v^\infty(\cdot) \in C^1_{\rm unif}(\mathbb{R},\mathbb{R})$, and $a^\infty_{j,k}(t,\boldsymbol{y})$, $b^\infty_j(t,\boldsymbol{y})$, $c^\infty_j(t,\boldsymbol{y})$, $c^\infty_0(t)$, $e^\infty_j(t,\boldsymbol{y})$ such that

$$a^\infty_{j,k}, b^\infty_j, c^\infty_j, c^\infty_0, e^\infty_j \quad \text{satisfy} \quad (H1),(H2),(H3) \tag{22}$$

with $t \ge 0$ in (H1), (H2) being replaced by $t \in \mathbb{R}$ and $[0,\infty)$ in (H3) being replaced by $(-\infty,\infty)$,

$$\lim_{s\to\infty} \|u^*(\cdot) - u^\infty(\cdot)\|_{C^1([s,\infty))} = 0, \tag{23}$$

$$\lim_{s\to\infty} \|v^*(\cdot) - v^\infty(\cdot)\|_{C^1([s,\infty))} = 0 \tag{24}$$

and

$$\lim_{s\to\infty} \|a_{j,k}(\cdot,\cdot) - a^\infty_{j,k}(\cdot,\cdot)\|_{C^1([s,\infty)\times\Omega_0)} = 0, \tag{25}$$

$$\lim_{s\to\infty} \|b_j(\cdot,\cdot) - b^\infty_j(\cdot,\cdot)\|_{C^1([s,\infty)\times\Omega_0)} = 0, \tag{26}$$

$$\lim_{s\to\infty} \|c_j(\cdot,\cdot) - c^\infty_j(\cdot,\cdot)\|_{C^1([s,\infty)\times\Omega_0)} = 0, \tag{27}$$

$$\lim_{s\to\infty} \|c_0(\cdot) - c^\infty_0(\cdot)\|_{C^1([s,\infty))} = 0, \tag{28}$$

$$\lim_{s\to\infty} \|e_j(\cdot,\cdot) - e^\infty_j(\cdot,\cdot)\|_{C^2([s,\infty)\times\partial\Omega_0)} = 0. \tag{29}$$

Then

$$\lim_{s\to\infty} \|f_u(u^*(\cdot),v^*(\cdot)) - f_u(u^\infty(\cdot),v^\infty(\cdot))\|_{C^1([s,\infty))} = 0,$$
$$\lim_{s\to\infty} \|f_v(u^*(\cdot),v^*(\cdot)) - f_v(u^\infty(\cdot),v^\infty(\cdot))\|_{C^1([s,\infty))} = 0,$$
$$\lim_{s\to\infty} \|g_u(u^*(\cdot),v^*(\cdot)) - g_u(u^\infty(\cdot),v^\infty(\cdot))\|_{C^1([s,\infty))} = 0,$$

and

$$\lim_{s\to\infty} \|g_v(u^*(\cdot),v^*(\cdot)) - g_v(u^\infty(\cdot),v^\infty(\cdot))\|_{C^1([s,\infty))} = 0.$$

Consider

$$\begin{cases} \frac{\partial \hat u}{\partial t} = \sum_{j,k=1}^n a^\infty_{j,k}(t,\boldsymbol{y}) \frac{\partial^2 \hat u}{\partial y_j \partial y_k} + \sum_{j=1}^n \big(b^\infty_j(t,\boldsymbol{y}) - c^\infty_j(t,\boldsymbol{y})\big)\frac{\partial \hat u}{\partial y_j} \\ \qquad + (c^\infty_0(t) + \gamma f_u(u^\infty(t),v^\infty(t)))\hat u + \gamma f_v(u^\infty(t),v^\infty(t))\hat v, \quad \boldsymbol{y}\in\Omega_0,\ t>s \\ \\ \frac{\partial \hat v}{\partial t} = d_c \sum_{j,k=1}^n a^\infty_{j,k}(t,\boldsymbol{y}) \frac{\partial^2 \hat v}{\partial y_j \partial y_k} + \sum_{j=1}^n \big(d_c\, b^\infty_j(t,\boldsymbol{y}) - c^\infty_j(t,\boldsymbol{y})\big)\frac{\partial \hat v}{\partial y_j} \\ \qquad + \gamma g_u(u^\infty(t),v^\infty(t))\hat u + (c^\infty_0(t) + \gamma g_v(u^\infty(t),v^\infty(t)))\hat v, \quad \boldsymbol{y}\in\Omega_0,\ t>s \\ \\ \sum_{j=1}^n e^\infty_j(t,\boldsymbol{y})\frac{\partial \hat u}{\partial y_j} = \sum_{j=1}^n e^\infty_j(t,\boldsymbol{y})\frac{\partial \hat v}{\partial y_j} = 0, \quad \boldsymbol{y}\in\partial\Omega_0,\ t>s \end{cases} \tag{30}$$

and

$$\begin{cases} \frac{d\hat u}{dt} = (c^\infty_0(t) + \gamma f_u(u^\infty(t),v^\infty(t)))\hat u + \gamma f_v(u^\infty(t),v^\infty(t))\hat v, \quad t>s \\ \\ \frac{d\hat v}{dt} = \gamma g_u(u^\infty(t),v^\infty(t))\hat u + (c^\infty_0(t) + \gamma g_v(u^\infty(t),v^\infty(t)))\hat v, \quad t>s, \end{cases} \tag{31}$$

where $s \in \mathbb{R}$. Let

$$\Phi^\infty(t,s)(u_0,v_0) = (\hat u(t,\cdot;s,u_0,v_0), \hat v(t,\cdot;s,u_0,v_0))$$

where $(\hat{u}(t,\cdot;s,u_0,v_0),\hat{v}(t,\cdot;s,u_0,v_0))$ is the solution of (30) with $(\hat{u}(s,\cdot;s,u_0,v_0),\hat{v}(s,\cdot;s,u_0,v_0)) = (u_0(\cdot),v_0(\cdot))\in X$, and let

$$\phi^\infty(t,s)(u_0,v_0) = (\hat{u}(t;s,u_0,v_0),\hat{v}(t;s,u_0,v_0))$$

where $(\hat{u}(t;s,u_0,v_0),\hat{v}(t;s,u_0,v_0))$ is the solution of (31) with $(\hat{u}(s;s,u_0,v_0),\hat{v}(s;s,u_0,v_0)) = (u_0,v_0)\in\mathbb{R}^2$.

Let

$$\lambda_+^\infty = \limsup_{t-s\to\infty,s\to\infty}\frac{\ln\|\phi^\infty(t,s)\|}{t-s},$$
$$\Lambda_+^\infty = \limsup_{t-s\to\infty,s\to\infty}\frac{\ln\|\Phi^\infty(t,s)\|}{t-s},$$

and

$$\lambda^\infty = \limsup_{t-s\to\infty}\frac{\ln\|\phi^\infty(t,s)\|}{t-s}$$

and

$$\Lambda^\infty = \limsup_{t-s\to\infty}\frac{\ln\|\Phi^\infty(t,s)\|}{t-s}.$$

λ_+^∞ and Λ_+^∞ are the corresponding limiting *forward top Lyapunov exponent* of (31) and (30), respectively, and λ^∞ and Λ^∞ are the corresponding limiting *top Lyapunov exponent* of (31) and (30), respectively.

Theorem 3.5. *Assume (H1)-(H5) and* (22)-(29)*.*

(1) *If Turing diffusion-driven instability for* (6) *occurs near* $(u^*(t),v^*(t))$*, then* $\lambda_+^\infty<0$ *and* $\Lambda_+^\infty\ge 0$*.*

(2) *If* $\lambda_+^\infty<0$ *and* $\Lambda_+^\infty>0$*, then Turing diffusion-driven instability for* (6) *occurs near* $(u^*(t),v^*(t))$*. In particular, if* $\{\phi^\infty(t,s)\}_{t\ge s}$ *admits a trivial exponential dichotomy and* $\{e^{-\lambda_0(t-s)}\Phi^\infty(t,s)\}_{t\ge s}$ *admits a nontrivial exponential dichotomy for some* $\lambda_0\ge 0$*, then Turing diffusion-driven instability for* (6) *occurs near* $(u^*(t),v^*(t))$*.*

Proof. (1) By Theorem 3.4 (1), we have

$$\lambda_+<0,\quad \Lambda_+\ge 0.$$

We first prove that $\Lambda_+^\infty\ge 0$. Assume that $\Lambda_+^\infty<0$. Then for given $\epsilon>0$ with $\Lambda_+^\infty+\epsilon<0$, there is $S>0$ such that

$$\frac{\ln\|\Phi^\infty(t,s)\|}{t-s}\le \Lambda_+^\infty+\epsilon<0\quad \forall\, t-s\ge S,\ s\ge S$$

and hence

$$\|\Phi^\infty(t,s)\|\le e^{(\Lambda_+^\infty+\epsilon)(t-s)}\quad \forall\, t-s\ge S,\ s\ge S.$$

Let

$$M=\big(1+\sup_{0\le t-s\le S,s\ge S}\|\Phi^\infty(t,s)\|\big)e^{-(\Lambda_+^\infty+\epsilon)S}.$$

Then

$$\|\Phi^\infty(t,s)\|\le Me^{(\Lambda_+^\infty+\epsilon)(t-s)} \tag{32}$$

for $t\ge s\ge S$.

Define $\{\tilde\Phi^\infty(t,s)\}_{t\ge s}$ by

$$\tilde\Phi^\infty(t,s)=\begin{cases}\Phi^\infty(t,s) & \text{if}\quad t\ge s\ge S,\\ \Phi^\infty(t,S)e^{(\Lambda_+^\infty+\epsilon)(S-s)} & \text{if}\quad t\ge S>s,\\ e^{(\Lambda_+^\infty+\epsilon)(t-s)}\cdot \mathrm{Id} & \text{if}\quad S>t\ge s.\end{cases}$$

Then $\{\tilde{\Phi}^\infty(t,s)\}_{t\ge s}$ is an exponentially bounded evolution family on X. By (32),

$$\|\tilde{\Phi}^\infty(t,s)\| \le Me^{(\Lambda_+^\infty+\epsilon)(t-s)} \tag{33}$$

for $t \ge s$.

By Theorem 2.3 (2),

$$\|\tilde{\Phi}^\infty(t+s,s) - \Phi(t+s,s)\| \to 0 \tag{34}$$

as $s \to \infty$ uniformly for t in compact subsets of $(0,\infty)$. For any given $S^* > 0$, let

$$\tilde{\Phi}(t,s;S^*) = \begin{cases} \Phi(t,s) & \text{if} \quad t \ge s \ge S^*, \\ \Phi(t,S^*) \circ \tilde{\Phi}^\infty(S^*,s) & \text{if} \quad t \ge S^* > s, \\ \tilde{\Phi}^\infty(t,s) & \text{if} \quad S^* > t \ge s. \end{cases}$$

By Theorem 2.3 (1), for any $\delta_n \ge 0$ with $\delta_n \to 0$ as $n \to \infty$ and any $S_n \to \infty$ as $n \to \infty$, one has

$$\begin{aligned} &\|\Phi(S_n+\delta_n,S_n)\tilde{\Phi}^\infty(S_n,S_n+\delta_n-1) - \tilde{\Phi}^\infty(S_n+\delta_n,S_n+\delta_n-1)\| \\ &= \|\big(\Phi(S_n+\delta_n,S_n) - \Phi^\infty(S_n+\delta_n,S_n)\big)\Phi^\infty(S_n,S_n+\delta_n-1)\| \\ &\to 0 \end{aligned} \tag{35}$$

as $n \to \infty$. (34) and (35) imply that

$$\|\tilde{\Phi}^\infty(1+s,s) - \tilde{\Phi}(1+s,s;S^*)\| \to 0$$

as $S^* \to \infty$ uniformly in $s \in \mathbb{R}$. By Theorem 2.11, there are $\tilde{S}^* > 0$, $\tilde{M} > 0$ and $\tilde{\lambda} < 0$ such that

$$\|\tilde{\Phi}(t,s;S^*)\| \le \tilde{M}e^{\tilde{\lambda}(t-s)}$$

for $t \ge s$. This implies that

$$\Lambda_+ = \limsup_{t-s\to\infty, s\to\infty} \frac{\ln\|\Phi(t,s)\|}{t-s} \le \tilde{\lambda} < 0,$$

which contradicts with $\Lambda_+ \ge 0$. Therefore, $\Lambda_+^\infty \ge 0$.

Next, we prove that $\lambda_+^\infty < 0$. It follows by similar arguments as above (λ_+ plays the role of Λ_+^∞ and λ_+^∞ plays that of Λ_+).

(2) First assume that $\lambda_+^\infty < 0$ and $\Lambda_+^\infty > 0$. By the similar arguments as in (1), we have $\lambda_+ < 0$. We claim that $\Lambda_+ > 0$. For otherwise, if $\Lambda_+ \le 0$, define $\tilde{\Phi}(t,s)$ and $\tilde{\Phi}^\infty(t,s)$ by

$$\tilde{\Phi}(t,s) = \Phi(t,s)e^{-\epsilon(t-s)} \quad \text{for} \quad t \ge s \ge 0$$

and

$$\tilde{\Phi}^\infty(t,s) = \Phi^\infty(t,s)e^{-\epsilon(t-s)} \quad \text{for} \quad t \ge s,$$

where $\epsilon = \frac{\Lambda_+^\infty}{2}$. Then

$$\tilde{\Lambda}_+ := \limsup_{t-s\to\infty, s\to\infty} \frac{\ln\|\tilde{\Phi}(t,s)\|}{t-s} \le -\epsilon < 0.$$

Note that

$$\|\tilde{\Phi}(1+s,s) - \tilde{\Phi}^\infty(1+s,s)\| \to 0$$

as $s \to \infty$. Again, by similar arguments as in (1), we have

$$\tilde{\Lambda}_+^\infty := \limsup_{t-s\to\infty, s\to\infty} \frac{\ln\|\tilde{\Phi}^\infty(t,s)\|}{t-s} = \Lambda_+^\infty - \epsilon = \frac{\Lambda_+^\infty}{2} < 0.$$

This is a contradiction. Therefore $\Lambda_+ > 0$. By Theorem 3.4 (2), Turing diffusion-driven instability for (6) occurs near $(u^*(t), v^*(t))$.

Next, assume that $\{\phi^\infty(t,s)\}_{t\ge s}$ admits a trivial exponential dichotomy and $\{e^{-\lambda_0(t-s)}\Phi^\infty(t,s)\}_{t\ge s}$ admits a nontrivial exponential dichotomy for some $\lambda_0 \ge 0$. Then we have $\lambda_+^\infty < 0$ and $-\lambda_0 + \Lambda_-^\infty > 0$ (hence $\Lambda_+^\infty > 0$). It then follows from the above arguments that $\lambda_+ < 0$ and $\Lambda_+ > 0$. By Theorem 3.4 (2), Turing diffusion-driven instability for (6) occurs near $(u^*(t), v^*(t))$. □

Observe that if (30) and (31) are periodic in t, then the limits $\lim_{t-s\to\infty} \frac{\ln \|\Phi^\infty(t,s)\|}{t-s}$, $\lim_{t-s\to\infty} \frac{\ln \|\phi^\infty(t,s)\|}{t-s}$ exist, and $\lambda^\infty = \lambda_+^\infty = \lim_{t-s\to\infty} \frac{\ln \|\phi^\infty(t,s)\|}{t-s}$, $\Lambda^\infty = \Lambda_+^\infty = \lim_{t-s\to\infty} \frac{\ln \|\Phi^\infty(t,s)\|}{t-s}$. In that case, we have the following corollary.

Corollary 1. *Assume (H1)-(H5) and* (22)-(29)*. Assume also that the limit systems* (30) *and* (31) *are periodic in t with period T. Then*

(1) If Turing diffusion-driven instability for (6) *occurs near* $(u^*(t), v^*(t))$*, then*

$$\lambda^\infty < 0, \quad \Lambda^\infty \ge 0.$$

(2) If

$$\lambda^\infty < 0, \quad \Lambda^\infty > 0,$$

then Turing diffusion-driven instability for (6) *occurs near* $(u^*(t), v^*(t))$.

The next section is devoted to a discussion of the case where the limit system (30) is additionally spatially homogeneous.

Remark 3. By using Lyapunov exponents, we have characterised Turing diffusion-driven instability for time-dependent domains. However, the explicit dependence of Lyapunov exponents on model parameters is very difficult to find in general. Classical Turing conditions in terms of model parameters cannot be expected in a general setting as outlined in the introduction.

4. **Applications.**

In this section, we discuss the application of the results established in Section 3 to the cases that the reaction-diffusion equations on the evolving domains have convection-free and spatially homogeneous limit equations.

Consider (6). Assume that there are $u^\infty(\cdot)$, $v^\infty(\cdot) \in C^1_{\rm unif}(\mathbb{R}, \mathbb{R})$, and $a^\infty_{j,k}(t, \boldsymbol{y})$, $b^\infty_j(t, \boldsymbol{y})$, $c^\infty_j(t, \boldsymbol{y})$, $c^\infty_0(t)$, $e^\infty_j(t, \boldsymbol{y})$ satisfying (22)-(29). Furthermore, assume

$$\begin{cases} a^\infty_{j,k}(t, \boldsymbol{y}) \equiv a_\infty(t)\delta_{jk}, \\ b^\infty_j(t, \boldsymbol{y}) \equiv 0, \\ c^\infty_j(t, \boldsymbol{y}) \equiv 0, \\ e^\infty_j(t, \boldsymbol{y}) \equiv e_\infty(t)\hat{\nu}_j(y). \end{cases} \tag{36}$$

Then (30) becomes

$$\begin{cases} \frac{\partial \hat{u}}{\partial t} = a_\infty(t)\Delta\hat{u} + (c^\infty_0(t) + \gamma f_u(u^\infty(t), v^\infty(t)))\hat{u} + \gamma f_v(u^\infty(t), v^\infty(t))\hat{v}, \\ \frac{\partial \hat{v}}{\partial t} = d_c a_\infty(t)\Delta\hat{v} + \gamma g_u(u^\infty(t), v^\infty(t))\hat{u} + (c^\infty_0(t) + \gamma g_v(u^\infty(t), v^\infty(t)))\hat{v}, \\ \sum_{j=1}^n \hat{\nu}_j(y)\frac{\partial \hat{u}}{\partial y_j} = \sum_{j=1}^n \hat{\nu}_j(y)\frac{\partial \hat{v}}{\partial y_j} = 0, \quad \boldsymbol{y} \in \partial\Omega_0,\ t > s. \end{cases} \tag{37}$$

Let λ_k and $\psi_k(\boldsymbol{y})$ $(k=0,1,2,\cdots)$ be the eigenvalues and eigenfunctions of

$$\begin{cases} \nabla^2\psi_k = -\lambda_k\psi_k, \;\; \boldsymbol{y}\in\Omega_0, \\ \\ (\boldsymbol{n}\cdot\nabla\psi_k)=0, \;\; \boldsymbol{y}\in\partial\Omega_0, \end{cases} \tag{38}$$

where $\lambda_0=0$ and $\lambda_k>0$ for $k\geq 1$. For given solution $(\hat{u}(t,y),\hat{v}(t,y))$ of (37), let

$$(\hat{u}(t,y),\hat{v}(t,y)) = \sum_{k=0}^{\infty}\psi_k(\boldsymbol{y})\left(\hat{u}_k(t),\hat{v}_k(t)\right). \tag{39}$$

Then $(\hat{u}_0(t),\hat{v}_0(t))$ satisfies

$$\begin{cases} \frac{d\hat{u}_0}{dt} = (c_0^\infty(t)+\gamma f_u(u^\infty(t),v^\infty(t)))\hat{u}_0+\gamma f_v(u^\infty(t),v^\infty(t))\hat{v}_0, \quad t>s, \\ \\ \frac{d\hat{v}_0}{dt} = \gamma g_u(u^\infty(t),v^\infty(t))\hat{u}_0+(c_0^\infty(t)+\gamma g_v(u^\infty(t),v^\infty(t)))\hat{v}_0, \quad t>s, \end{cases} \tag{40}$$

and $(\hat{u}_k(t),\hat{v}_k(t))$ satisfies

$$\begin{cases} \frac{d\hat{u}_k}{dt} = -a_\infty(t)\lambda_k\hat{u}_k+(c_0^\infty(t)+\gamma f_u(u^\infty(t),v^\infty(t)))\hat{u}_k+\gamma f_v(u^\infty(t),v^\infty(t)))\hat{v}_k, \\ \\ \frac{d\hat{v}_k}{dt} = -d_c a_\infty(t)\lambda_k\hat{v}_k+\gamma g_u(u^\infty(t),v^\infty(t))\hat{u}_k+(c_0^\infty(t)+\gamma g_v(u^\infty(t),v^\infty(t))))\hat{v}_k, \\ \qquad\qquad\qquad\qquad t>s, \end{cases} \tag{41}$$

for $k=1,2,\cdots$.

Let

$$\phi_0^\infty(t,s)(u,v) = (\hat{u}_0(t;s,u,v),\hat{v}_0(t;s,u,v))$$

where $(\hat{u}_0(t;s,u,v),\hat{v}_0(t;s,u,v))$ is the solution of (40) with $(\hat{u}_0(s;s,u,v),\hat{v}_0(s;s,u,v)) = (u,v)\in\mathbb{R}^2$ and let

$$\phi_k^\infty(t,s)(u,v) = (\hat{u}_k(t;s,u,v),\hat{v}_k(t;s,u,v))$$

where $(\hat{u}_k(t;s,u,v),\hat{v}_k(t;s,u,v))$ is the solution of (41) with $(\hat{u}_k(s;s,u,v),\hat{v}_k(s;s,u,v)) = (u,v)\in\mathbb{R}^2$. Let

$$\lambda_{0,+}^\infty = \limsup_{t-s\to\infty,s\to\infty}\frac{\ln\|\phi_0^\infty(t,s)\|}{t-s}$$

and

$$\lambda_{k,+}^\infty = \limsup_{t-s\to\infty,s\to\infty}\frac{\ln\|\phi_k^\infty(t,s)\|}{t-s}$$

be the forward top Lyapunov exponent of (40) and (41), respectively. Note that $\lambda_{0,+}^\infty=\lambda_+^\infty$.

Theorem 4.1. *Assume (H1)-(H5),* (22)-(29)*, and* (36).

1. *(1) If Turing diffusion-driven instability for* (6) *occurs near* $(u^*(t),v^*(t))$*, then* $\lambda_{0,+}^\infty<0$ *and* $\lambda_{k,+}^\infty\geq 0$ *for some* $k\geq 1$.
2. *(2) If* $\lambda_{0,+}^\infty<0$ *and* $\lambda_{k,+}^\infty>0$ *for some* $k\geq 1$*, then Turing diffusion-driven instability for* (6) *occurs near* $(u^*(t),v^*(t))$.

Proof. (1) By Theorem 3.5 (1), we have $\lambda_+^\infty < 0$ and $\Lambda_+^\infty \ge 0$. Hence $\lambda_{0,+}^\infty = \lambda_+^\infty < 0$. Next we prove that $\lambda_{k,+}^\infty \ge 0$ for some $k \ge 1$.

Observe that there is $M > 0$ such that for any given $k \ge 1$ and any solution $(\hat{u}_k(t), \hat{v}_k(t))$ of (41), the following hold

$$|\hat{u}_k(t)| \le e^{-\lambda_k \int_s^t a_\infty(r)dr} |\hat{u}_k(s)| + M \int_s^t e^{-\lambda_k \int_\tau^t a_\infty(r)dr} \big(|\hat{u}_k(\tau)| + |\hat{v}_k(\tau)|\big) d\tau$$

and

$$|\hat{v}_k(t)| \le e^{-d_c\lambda_k \int_s^t a_\infty(r)dr} |\hat{u}_k(s)| + M \int_s^t e^{-d_c\lambda_k \int_\tau^t a_\infty(r)dr} \big(|\hat{u}_k(\tau)| + |\hat{v}_k(\tau)|\big) d\tau$$

for all $t \ge s$. Hence

$$\begin{aligned}|\hat{u}_k(t)| + |\hat{v}_k(t)| \le e^{-d_c^*\lambda_k \int_s^t a_\infty(r)dr} \big(|\hat{u}_k(s)| + \\ + |\hat{u}_k(s)|\big) + M \int_s^t e^{-d_c^*\lambda_k \int_\tau^t a_\infty(r)dr} \big(|\hat{u}_k(\tau)| + |\hat{v}_k(\tau)|\big) d\tau\end{aligned}$$

for all $t \ge s$, where $d_c^* = \min\{1, d_c\}$. It then follows from Gronwall's inequality that

$$|\hat{u}_k(t)| + |\hat{v}_k(t)| \le e^{M(t-s) - d_c^*\lambda_k \int_s^t a_\infty(r)dr} \big(|\hat{u}_k(s)| + |\hat{u}_k(s)|\big) \tag{42}$$

for all $t \ge s$. Therefore there are $\lambda^* > 0$ and $S^*, K^* > 0$ such that for any $k \ge K^*$,

$$|\hat{u}_k(t)| + |\hat{v}_k(t)| \le e^{-\lambda^*(t-s)} \big(|\hat{u}_k(s)| + |\hat{u}_k(s)|\big) \quad \forall t \ge s + S^*. \tag{43}$$

Assume that $\lambda_{k,+}^\infty < 0$ for every $k \ge 1$. Then there are $\lambda^{**} > 0$ and $S^{**} > 0$ such that

$$|\hat{u}_k(t)| + |\hat{v}_k(t)| \le e^{-\lambda^{**}(t-s)} \big(|\hat{u}_k(s)| + |\hat{u}_k(s)|\big) \quad \forall t \ge s + S^{**} \tag{44}$$

for $1 \le k \le K^*$. By (43) and (44),

$$\Lambda_+^\infty \le -\min\{\lambda^*, \lambda^{**}\} < 0.$$

This is a contradiction. Therefore, $\lambda_{k,+}^\infty \ge 0$ for some $k \ge 1$.

(2) Note that $\Lambda_+^\infty \ge \lambda_{+,k}^\infty$ for any $k \ge 1$. If $\lambda_{0,+}^\infty < 0$ and $\lambda_{k,+}^\infty > 0$ for some $k \ge 1$, then $\lambda_+^\infty = \lambda_{0,+}^\infty < 0$ and $\Lambda_+^\infty > 0$. By Theorem 3.5 (2), Turing diffusion-driven instability for (6) occurs near $(u^*(t), v^*(t))$. □

In the following, we consider two special cases, that is, the case that the limit system is time-periodic and the case that the limit system is time-independent. First we assume that the limit system is time-periodic, that is, there is $T > 0$ such that

$$\begin{cases} a_\infty(t+T) = a_\infty(t), \\ c_0^\infty(t+T) = c_0^\infty(t), \\ (u_\infty(t+T), v_\infty(t+T)) = (u_\infty(t), v_\infty(t)). \end{cases} \tag{45}$$

Let $\mu_{0,1}$, $\mu_{0,2}$ be the eigenvalues of $\phi_0^\infty(T, 0)$ (i.e. Floquet or characteristic multipliers of (40)) and $\mu_{k,1}^\infty$ and $\mu_{k,2}^\infty$ be the eigenvalues of $\phi_k^\infty(T, 0)$ (i.e. Floquet or characteristic multipliers of (41)), respectively.

Corollary 2. *Assume the conditions in Theorem 4.1 and* (45)*.*

(1) If Turing diffusion-driven instability for (6) *occurs near* $(u^*(t), v^*(t))$*, then* $\max\{|\mu_{0,1}|, |\mu_{0,2}|\} < 1$ *and* $\max\{|\mu_{k,1}|, |\mu_{k,2}|\} \ge 1$ *for some* $k \ge 1$*.*

(2) If $\max\{|\mu_{0,1}|, |\mu_{0,2}|\} < 1$ *and* $\max\{|\mu_{k,1}|, |\mu_{k,2}|\} > 1$ *for some* $k \ge 1$*, then Turing diffusion-driven instability for* (6) *occurs near* $(u^*(t), v^*(t))$*.*

Proof. It follows from Theorem 4.1 and the fact that

$$\begin{cases} \lambda_0^+ = \frac{1}{T}\ln[\max\{|\mu_{0,1}|,|\mu_{0,2}|\}], \\ \lambda_k^+ = \frac{1}{T}\ln[\max\{|\mu_{k,1}|,|\mu_{k,2}|\}] \end{cases}$$

for $k = 1, 2, \cdots$. □

Next we assume that the limit system is time-independent and $c_0^\infty(t) \equiv 0$, that is,

$$\begin{cases} a_\infty(t) \equiv a_\infty, \\ c_0^\infty(t) \equiv 0, \\ (u_\infty(t), v_\infty(t)) \equiv (u_\infty, v_\infty). \end{cases} \tag{46}$$

Note that under the assumption (36) and $a_\infty(t) \equiv a_\infty$, $(u_\infty(t), v_\infty(t)) \equiv (u_\infty, v_\infty)$, it is natural to assume $c_0^\infty(t) \equiv 0$.

Remark 4. Conditions (46) hold for the case of linear uniform isotropic logistic growth [16].

Assuming that conditions (46) hold, we provide explicit necessary and sufficient conditions for the Turing diffusion-driven instability to occurs for (6) near $(u^*(t), v^*(t))$. To this end, let

$$b(\lambda) = a_\infty(1+d_c)\lambda - \gamma\left(f_u + g_v\right), \tag{47}$$

$$c(\lambda) = a_\infty^2 d_c \lambda^2 - \lambda\Big[\gamma a_\infty(d_c f_u + g_v)\Big] + \gamma^2(f_u g_v - f_v g_u). \tag{48}$$

Let

$$\begin{aligned} p_2 &= a_\infty^2 d_c > 0, \\ p_1 &= -\gamma a_\infty(d_c f_u + g_v), \\ p_0 &= c(0). \end{aligned} \tag{49}$$

In the above, the subscripts u, v denote partial differentiation and f_u, f_v, g_u and g_v are evaluated at (u_∞, v_∞). If $p_1 < 0$ and $p_1^2 - 4a_\infty^2 d_c c(0) > 0$, let $k_\pm^2$ be such that

$$k_\pm^2 = \frac{-p_1 \pm \sqrt{p_1^2 - 4a_\infty^2 d_c c(0)}}{2a_\infty^2 d_c}. \tag{50}$$

Corollary 3. *(1) Assuming conditions in Theorem 4.1 and* (46) *hold, the necessary conditions for a Turing diffusively-driven instability corresponding to the system* (6) *near* $(u^*(t), v^*(t))$ *are given by*

$$\gamma(f_u + g_v) < 0, \tag{51}$$

$$c(0) = \gamma^2(f_u\, g_v - f_v\, g_u) > 0, \tag{52}$$

$$-\frac{p_1}{a_\infty} = \gamma(d_c f_u + g_v) > 0, \tag{53}$$

$$\Big[\gamma(d_c f_u + g_v)\Big]^2 - 4d_c\Big[\gamma^2(f_u g_v - f_v g_u)\Big] > 0, \tag{54}$$

where the subscripts u, v *denote partial differentiation, with the Jacobian components and* f_u, f_v, g_u *and* g_v *are evaluated at* (u_∞, v_∞).

(2) In addition to conditions in (1), if there is some k *such that*

$$\lambda_k \in \left(k_-^2, k_+^2\right),$$

then Turing diffusively-driven instability (6) *near* $(u^*(t), v^*(t))$ *occurs.*

Proof. (1) Suppose that μ is an eigenvalue of the associated eigenvalue problem of (37) and $\sum_{k=0}^{\infty}\psi_k(\boldsymbol{y})\boldsymbol{w}_k$ is a corresponding eigenfunction. Then

$$(\hat{u},\hat{v}) = e^{\mu t}\sum_{k=0}^{\infty}\psi_k(\boldsymbol{y})\boldsymbol{w}_k \tag{55}$$

is a solution of (37). By (40) and (41), we have

$$\mu \boldsymbol{I}\boldsymbol{w}_k = -a_\infty\lambda_k\boldsymbol{D}\boldsymbol{w}_k + \boldsymbol{J}_F\boldsymbol{w}_k, \quad k \ge 0, \tag{56}$$

where $\boldsymbol{D} = \begin{pmatrix} 1 & 0 \\ 0 & d_\infty \end{pmatrix}$, and $\boldsymbol{I}$ is the identity matrix. If $\boldsymbol{w}_k$ is a non-zero vector, then

$$\begin{vmatrix} \mu + a_\infty\lambda_k - \gamma f_u & -\gamma f_v \\ -\gamma g_u & \mu + d_c a_\infty \lambda_k - \gamma f_u \end{vmatrix} = 0. \tag{57}$$

It can be shown that the characteristic equation is given by

$$\mu^2 + b(\lambda_k)\mu + c(\lambda_k) = 0. \tag{58}$$

Solving equation (58) yields $\mu = \mu_k^+$ or μ_k^-, where

$$2\mu_k^{\pm} = -b(\lambda_k) \pm \sqrt{b^2(\lambda_k) - 4c(\lambda_k)}. \tag{59}$$

By Theorem 4.1, if Turing diffusively-driven instability (6) near $(u^*(t), v^*(t))$ occurs, then $\mathrm{Re}\left(\mu_0^+\right) < 0$ and there is some $k \ge 1$ such that $\mathrm{Re}\left(\mu_k^+\right) \ge 0$.

Observe that in the absence of diffusion, i.e. when $k = 0$, $\mathrm{Re}\left(\mu_0^+\right) < 0$ if and only if

$$b(0) > 0 \Longrightarrow \gamma\left(f_u + g_v\right) < 0, \tag{60}$$

$$c(0) > 0 \Longrightarrow \gamma^2(f_u g_v - f_v g_u) > 0, \tag{61}$$

that is, (51) and (52) hold.

In the presence of diffusion we require that $\mathrm{Re}\left(\mu_k^+\right) \ge 0$ for some $k \ge 1$. It can be shown easily that the coefficient of μ in the characteristic equation (58) is positive and is given by

$$b(\lambda) = a_\infty(1 + d_c)\lambda_k + b(0) > 0.$$

Hence, if growth occurs, we must have $c(\lambda_k) \le 0$ for some $k \ge 1$. Expressing $c(\lambda_k)$ as a quadratic polynomial we have

$$c(\lambda_k) = p_2\lambda_k^2 + p_1\lambda_k + p_0 \tag{62}$$

We therefore require that

$$\gamma a_\infty(d_c f_u + g_v) > 0, \tag{63}$$

that is, (53) holds, to guarantee that $c(\lambda_k) \le 0$ for some $k \ge 1$. For diffusively-driven instability to occur, we also require that there exists real $k_{\pm}^2$ such that $c(k_{\pm}^2) = 0$ and these can be easily shown to be given by

$$k_{\pm}^2 = \frac{-p_1 \pm \sqrt{p_1^2 - 4a_\infty^2 d_c c(0)}}{2a_\infty^2 d_c}.$$

Thus, requiring $c(\lambda_k) \le 0$ entails $p_1^2 - 4a_\infty^2 d_c c(0) > 0$, thereby yielding the last condition, that is, (54), or for diffusively-driven instability given by

$$\Big[\gamma a_\infty(d_c f_u + g_v)\Big]^2 - 4a_\infty^2 d_c\Big[\gamma^2(f_u g_v - f_v g_u)\Big] > 0. \tag{64}$$

(2) Under the conditions in (1), $p_1 < 0$ and $p_1^2 - 4a_\infty^2 d_c c(0) > 0$. If there is $\lambda_k \in (k_-^2, k_+^2)$, then $c(\lambda_k) < 0$. This implies that there is $\mu_k > 0$ and $\boldsymbol{w}_k$ such that

$(\hat{u}, \hat{v}) = e^{\mu_k t}\psi_k(y)\boldsymbol{w}_k$ is a solution of (41). By Theorem 4.1, Turing diffusively-driven instability (6) near $(u^*(t), v^*(t))$ occurs. □

Remark 5. Note that the above conditions generalize the classic results for fixed domains [38, 10, 27] in the case of limiting systems. In addition, the inequalities (51)–(54) define a time-independent domain in parameter space, generalizing the Turing space.

Remark 6. It is not at all a trivial process to formulate Turing diffusion-driven instability conditions in terms of reaction kinetics and model parameter values, if time-dependent domains (periodic or otherwise) are considered and $c_0^\infty(t) \neq 0$ and $a_\infty(t) \neq 0$. This will be the subject of future research for scientifically relevant special cases.

5. **Conclusion and Discussions.** In biological pattern formation, the theoretical stability analysis of reaction-diffusion equations (RDEs) on continuously deforming domains and evolving surfaces has remained largely elusive. Despite a considerable amount of research in this area, progress has been limited to special types of growth evolution. For example, as the first step in considering the Turing diffusively-driven instability analysis on growing domains, the RDEs are transformed into RDEs on fixed domains, but with time-dependence in the diffusion and dilution terms [5, 29]. These nonautonomous terms however typically invalidate standard linear stability analysis via plane wave decompositions, even with the common simplification that the domain growth is assumed to be isotropic. In a recent paper by Madzvamuse *et al.* [16] theoretical stability analysis was successfully studied using asymptotic theory for the case of continuously deforming domains under the assumptions of slow growth. In that paper, *domain-induced* Turing diffusion-driven instability conditions were stated and proved when $c_0(t) \sim O(\epsilon)$ for slow, isotropic domain growth. In this paper we extend significantly these results by considering the long-time behaviour of the solutions of the RDEs. One might consider this as the limiting case for the asymptotic analysis.

By using the general evolution semigroup or evolution family theory [1, 4, 34], Turing diffusion-driven instability around a spatially homogeneous solution (time-dependent manifold) is characterized by Lyapunov exponents of the evolution family associated to the linearized system around the time-dependent manifold. This is the main result of our paper. In order to state and prove this result, we re-defined the concept of diffusion-driven instability on time-dependent domains in terms of Lyapunov exponents. Furthermore, we state precisely under what conditions our results can be reduced to those already known and derived on fixed domains where the eigenvalue theory is used. Our analysis allows for the inclusion to study limiting systems, i.e. when domain growth saturates to a final fixed domain. This scenario is biologically plausible since most species grow to a finite limiting size as opposed to an infinite domain size. For this case, we state and prove the corresponding diffusion-driven instability conditions. The key difference between these results and those obtained on fixed domain is that the diffusion coefficient is scaled by the limiting domain growth profile.

Our analysis identifies two important processes closely related which influence the type of diffusion-driven instability conditions one obtains. These are the diffusion of the chemical concentrations or molecules and the flow field of the evolving domain. For the case of linear isotropic growth of the domain, one of these terms vanishes and this is the case for linear, exponential and logistic growth functions. If the diffusion term is nonzero, and the divergence of the flow field is zero, we stated and proved the conditions for diffusion-driven instability and these are similar to those obtained on a fixed domain. On the other hand, assuming that both terms are nonzero, the conditions for diffusion-driven instability may depend on the growth profile as well as the kinetic parameter values. Although we have not been able to find a biological example when both terms are nonzero, these conditions have been shown to hold when domain growth is slow, linear and isotropic [16]. Under these assumptions, the need for *short-range activation and long-range inhibition* can be relaxed. Hence a wider range of biological morphogen pairings have the potential to induce Turing patterning on a growing domain compared to a fixed domain. It is now possible to suggest and investigate, for example, *activator-activator* or *short-range inhibition, long-range activation* as paradigms for biological pattern formation on growing domains.

In heart physiology, the domain changes periodically. We have extended Turing diffusion-driven instability analysis to periodic continuously deforming domains, with period say, T. For this case, we state under what conditions diffusion-driven instability occurs. To our knowledge, this is the first time such a result has been stated and proved.

This paper has established theoretical foundations to carry out numerical studies and computational experiments on domains which either saturate to a final limiting fixed size or grow and contract periodically. In developmental biology, applications of this theory could be used to study realistic growth profiles whose growth functions possess a finite limit or have periodic behaviour.

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