Saddle Point Problems and Mixed Formutaions

Applications in Fluids, Elasticity, and Poroelasticity

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Outline

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Some P.D.E.

- Poisson's equation $-\Delta u = f$ in Ω
- Helmholtz equation $-\Delta u + ru = f$ in Ω
- Heat equation $u_t - \Delta u = f$ in $\Omega \times (0, T)$
- Wave equation $u_{tt} - \Delta u = f$ in $\Omega \times (0, T)$

The domain $\Omega \subset \mathbb{R}^d$ with $d \geq 2$, which is of class $C^{0,1}$, with boundary $\partial\Omega$

$-\Delta u = -\sum_{i=1}^{d} u_{x_i x_i}$

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$-\Delta u = -\sum_{i=1}^{d} u_{x_i x_i}$

Elliptic P.D.E.

Model Equation

Consider

$$-\nabla \cdot (k\nabla u) + ru = f \qquad \text{in } \Omega$$

where k is positive (bounded away from zero) and r is nonnegative

---

$-\Delta u = -\nabla \cdot \nabla u$

Elliptic P.D.E.

Boundary Conditions

Dirichlet type

$$u|_{\partial\Omega} = g$$

Neumann type

$$-k\nabla u \cdot \mathbf{n}|_{\partial\Omega} = h$$

Robin type

$$-k\nabla u \cdot \mathbf{n}|_{\partial\Omega} = \gamma(u_s - u|_{\partial\Omega})$$

One can also have mixed boundary conditions

Elliptic P.D.E.
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One can also have mixed boundary conditions

Weak Formulation

To derive a weak formulation for

$$-\nabla \cdot (k\nabla u) + ru = f \qquad \text{in } \Omega$$

multiply the equation by a test function v and integrate over Ω (integrating by parts)

$$\begin{aligned}\int_\Omega [-\nabla \cdot (k\nabla u) + ru]v\,dx &= -\int_{\partial\Omega} k\nabla u \cdot \mathbf{n} v\,ds \\ &\quad + \int_\Omega k\nabla u \cdot \nabla v + ruv\,dx \\ &= \int_\Omega fv\,dx\end{aligned}$$

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To derive a weak formulation for

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&\quad + \int_\Omega k\nabla u \cdot \nabla v + ruv \, dx \\
&= \int_\Omega fv \, dx
\end{aligned}$$

Weak Formulation

Notation

Set

$$a(u, v) = \int_\Omega k\nabla u \nabla v + ruv\, dx$$

$$(u, v) = \int_\Omega uv\, dx$$

$$\langle f, v\rangle_\Omega = \int_\Omega fv\, dx$$

$$\langle u, v\rangle_{\partial\Omega} = \int_{\partial\Omega} uv\, ds$$

Weak Formulation

Spaces

$L^2(\Omega)$ - space of functions which are square integrable on Ω

$$\int_\Omega |u|^2 \, dx < \infty$$

with inner product (u, v) and norm

$$\|v\|_0 = \sqrt{(v, v)}$$

Weak Formulation

Spaces

$H^1(\Omega)$ - space of square integrable functions on Ω with first weak derivatives that are square integrable

$$H^1(\Omega) = \{v \in L^2(\Omega) : \nabla v \in L^2(\Omega)\}$$

with inner product $(\nabla u, \nabla v) + (u, v)$ and norm

$$\|v\|_1 = \sqrt{(\nabla v, \nabla v) + (v, v)}$$

Weak Formulation

Spaces

$H_0^1(\Omega)$ - space of $H^1(\Omega)$ functions that have trace zero on the boundary

$$H_0^1(\Omega) = \{v \in H_0^1(\Omega) : u|_{\partial\Omega} = 0\}$$

$H^{1/2}(\partial\Omega)$ - is the trace space of $H^1(\Omega)$ with dual $H^{-1/2}(\partial\Omega)$

$H^{-1}(\Omega)$ - is the dual of $H_0^1(\Omega)$

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The Dirichlet Problem

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$$u|_{\partial\Omega} = 0 \qquad \text{on } \partial\Omega$$

Given $f \in H^{-1}(\Omega)$ find $u \in H_0^1(\Omega)$ such that

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$$-\nabla \cdot (k\nabla u) + ru = f \qquad \text{in } \Omega$$

$$-k\nabla u \cdot \mathbf{n}|_{\partial\Omega} = h \qquad \text{on } \partial\Omega$$

Given $f \in H^*(\Omega)$ and $h \in H^{-1/2}(\partial\Omega)$ find $u \in H^1(\Omega)$ such that

$$a(u, v) = \langle f, v\rangle_\Omega - \langle h, v\rangle_{\partial\Omega} \qquad \text{for all } v \in H^1(\Omega)$$

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Abstract Setting

The Lax Milgram Lemma

V - a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$

$a : V \times V \mapsto \mathbb{R}$ a continuous bilinear form, i.e., there exists a number M such that

$$|a(u, v)| \leq M\|u\|_V \|v\|_V \qquad \text{for all } u, v \in V$$

a is V-elliptic, i.e., there exists a number $\alpha > 0$ such that

$$\alpha\|u\|_V^2 \leq a(u, u) \qquad \text{for all } u \in V$$

$F : V \mapsto \mathbb{R}$ is a continuous linear form, i.e.

$$|F(v)| \leq \|F\|_{V^*} \|v\|_V$$

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Then, find $u \in V$ such that

$$a(u, v) = F(v) \qquad \text{for all } v \in V$$

has a unique solution, furthermore the solution satisfies the following a priori estimate

$$\|u\|_V \leq \frac{1}{\alpha}\|F\|_{V^*}$$

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U - a Banach space

V - a reflexive Banach space

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a satisfies the $\inf - \sup$ condition, i.e., there exists a number $\alpha > 0$ such that

$$\inf_{u \in U} \sup_{v \in V} \frac{a(u,v)}{\|u\|_U \|v\|_V} \geq \alpha$$

$$a(u,v) = 0 \quad \text{for all } u \in U \implies v = 0$$

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Variational Problem

The solution of the problem find $u \in V$ such that

$$a(u, v) = F(v) \qquad \text{for all } v \in V$$

is the minimizer of

$$J(u) = \frac{1}{2}a(u, u) - F(u)$$

Constrained Variational Problem

The solution of the problem find $u \in V$ such that

$$a(u, v) = F(v) \qquad \text{for all } v \in V$$

subject to the constraint

$$b(u, \mu) = G(\mu) \qquad \text{for all } \mu \in M$$

is the saddle point of the Lagrange function

$$L(u, \lambda) = J(u) + [b(u, \lambda) - G(\lambda)]$$

Mixed Problem

Ladyzhenskaya Babuska Brezzi

Find $u \in V$ and $\lambda \in M$ such that

$$\begin{aligned} a(u,v) + b(v,\lambda) &= F(v) && \text{for all } v \in V \\ b(u,\mu) &= G(\mu) && \text{for all } \mu \in M \end{aligned}$$

Solutions satisfy the saddle property

$$L(u,\mu) \leq L(u,\lambda) \leq L(v,\lambda) \qquad \text{for all } v \in V, \mu \in M$$

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Assume the forms a and b are continuous, i.e., there exists numbers A and B such that

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$$|b(v,\mu)| \leq B\|u\|_V\|\mu\|_M \qquad \text{for all } v \in V, \mu \in M$$

Assume the functionals F and G are bounded, i.e.,

$$|F(v)| \leq \|F\|_{V^*}\|v\|_V \qquad \text{for all } v \in V$$

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$$\alpha \|u\|_V^2 \leq a(u,u) \qquad \text{for all } u \in V \quad \text{with } b(u,\mu) = 0, \qquad \text{for all } \mu \in M$$

and that b satisfies the $\inf - \sup$ condition, i.e., there exists a $beta > 0$

$$\inf_{\mu \in M} \sup_{v \in V} \frac{b(v,\mu)}{\|v\|_V \|\mu\|_M} \geq \beta$$

Then the mixed problem has a unique solution

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Examples

Poisson's Equation Primal Formulation

$$-\Delta u = -\nabla \cdot \nabla u = f$$

can be written as

$$\begin{aligned} \nabla u &= \sigma \\ -\nabla \cdot \sigma &= f \end{aligned}$$

This yields the saddle point problem, find $(\sigma, u) \in (L^2(\Omega))^d \times H_0^1(\Omega)$ such that

$$\begin{aligned} (\sigma, \tau) - (\tau, \nabla u) &= 0 && \text{for all } \tau \in (L^2(\Omega))^d \\ -(\sigma, \nabla v) &= -(f, v) && \text{for all } v \in H_0^1(\Omega) \end{aligned}$$

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This yields the saddle point problem, find $(\sigma, u) \in H(\text{div}, \Omega) \times L^2(\Omega)$ such that

$$\begin{aligned} (\sigma, \tau) + (\nabla \cdot \tau, u) &= 0 && \text{for all } \tau \in H(\text{div}, \Omega) \\ (\nabla \cdot \sigma, v) &= -(f, v) && \text{for all } v \in L^2(\Omega) \end{aligned}$$

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Examples
Stokes Equations

$$-\Delta u - \nabla p = f \qquad \text{in } \Omega$$

$$\nabla \cdot u = 0 \qquad \text{in } \Omega$$

$$u\big|_{\partial\Omega} = 0 \qquad \text{on } \partial\Omega$$

Find $(u, p) \in (H_0^1(\Omega))^d \times L_0^2(\Omega)$ such that

$$\begin{aligned}(\nabla u : \nabla v) + (\nabla \cdot v, p) &= (f, v) \qquad \text{for all } v \in (H_0^1(\Omega))^d \\ (\nabla \cdot u, q) &= 0 \qquad \text{for all } q \in L_0^2(\Omega)\end{aligned}$$

where

$$L_0^2(\Omega) = \{q \in L^2(\Omega) : (q, 1) = 0\}$$

Examples

Stokes Equations

$$-\Delta u - \nabla p = f \qquad \text{in } \Omega$$

$$\nabla \cdot u = 0 \qquad \text{in } \Omega$$

$$u|_{\partial\Omega} = 0 \qquad \text{on } \partial\Omega$$

Find $(u, p) \in (H_0^1(\Omega))^d \times L_0^2(\Omega)$ such that

$$\begin{aligned} (\nabla u : \nabla v) + (\nabla \cdot v, p) &= (f, v) \qquad \text{for all } v \in (H_0^1(\Omega))^d \\ (\nabla \cdot u, q) &= 0 \qquad \text{for all } q \in L_0^2(\Omega) \end{aligned}$$

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Examples

Navier Stokes Equations

$$-\Delta u + u \cdot \nabla u - \nabla p = f \qquad \text{in } \Omega$$

$$\nabla \cdot u = 0 \qquad \text{in } \Omega$$

$$u|_{\partial\Omega} = 0 \qquad \text{on } \partial\Omega$$

Find $(u, p) \in (H_0^1(\Omega))^d \times L_0^2(\Omega)$ such that

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Navier Stokes Equations

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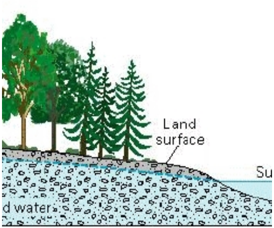
Porormechanics

Poromechanics is the science of energy, motion, and forces and their effect on porous material and in particular the mechanical behavior (swelling and shrinking) of fluid-saturated porous media. Poromechanics and electro-poromechanics are complex coupled, multiscale, phenomena, where the swelling and shrinking of an elastic (or viscoelastic, viscoplastic, or plastic) deforming porous medium is coupled to the electro-chemo-thermo-mechanical response of the medium and the fluid.

Porormechanics

Modeling and predicting the mechanical (or the electro-chemo-thermo-mechanical) behavior of fluid-infiltrated porous media is of great importance since many natural substances, e.g., rocks, soils, and biological tissues, as well as man made materials such as foams, gels, concrete, and ceramics can be considered as elastic porous media.



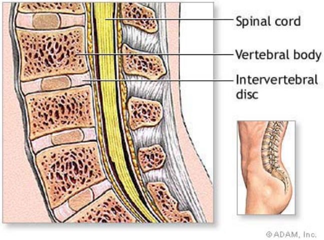
Application Areas

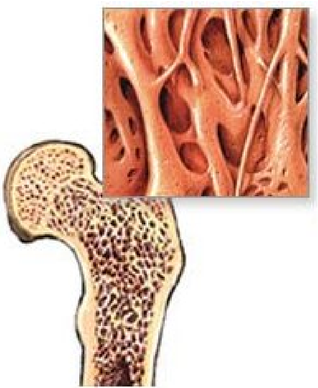
Geomechanics

Clays, shales, damage shrinkage of concrete

Application Areas
Biomechanics

Hydrated tissues, bones, corneal swelling, hydrogels, intervertebral discs

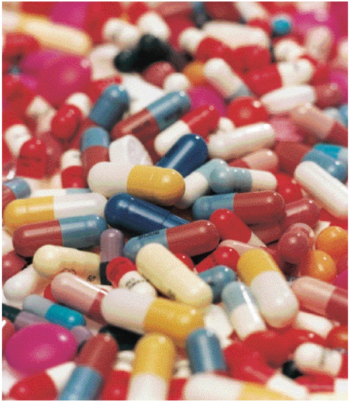


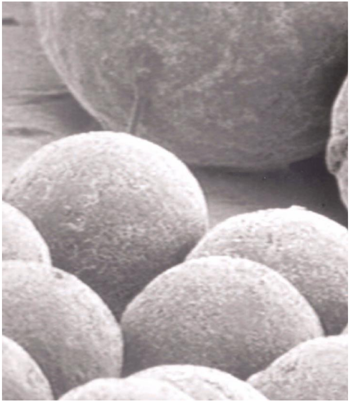


Application Areas

Pharmacology

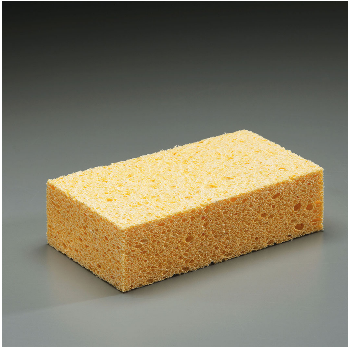
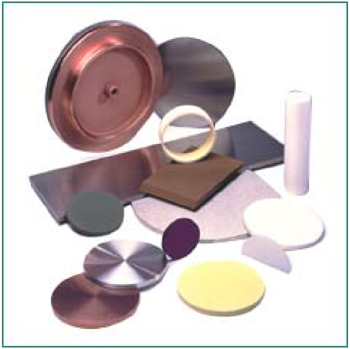
Water-solute drug carriers, biodegradable drug delivery-systems





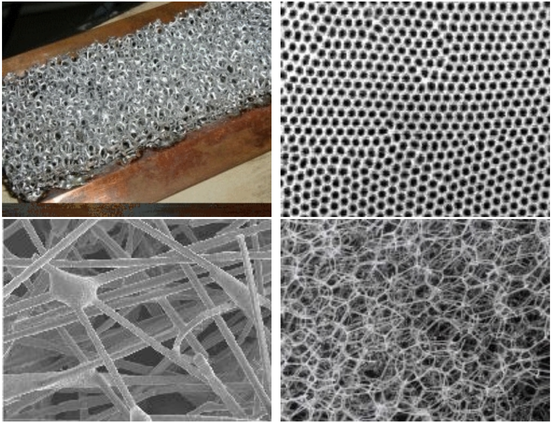
Application Areas
Material Science

Polymeric materials, crosslinked porous structures, foams, gels, and ceramics

Application Areas
Material Science

High-tech material



Models

- Rigid — Non-rigid
- Saturated — Unsaturated
- Fully Dynamic — Quasistatic — Steady
- Incompressible — Slightly Compressible
- Secondary Consolidation

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Mathematical Model

Basic Quantities

Stress

$$\begin{aligned}\tau &= \mu\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right) + \left(\lambda^*\frac{\partial}{\partial t}\nabla\cdot\mathbf{u} + \lambda\nabla\cdot\mathbf{u} - \alpha p\right) I \\ &= 2\mu\varepsilon(\mathbf{u}) + \left(\lambda^*\frac{\partial}{\partial t}\nabla\cdot\mathbf{u} + \lambda\nabla\cdot\mathbf{u} - \alpha p\right) I\end{aligned}$$

$\mathbf{u}$ displacement

p pressure

μ λ Lamé coefficients

λ^* coefficient of secondary consolidation

α Biot-Willis constant (couples pressure and deformation)

ε Strain

Mathematical Model

Basic Quantities

Fluid content

$$\eta = c_0 p + \alpha \nabla \cdot \mathbf{u}$$

Fluid flux, Darcy's law

$$\mathbf{q} = -\kappa \nabla p$$

c_0 combined porosity and compressibility
κ hydraulic conductivity

Mathematical Model

Conservation Laws

Balance of momentum

$$\rho \frac{\partial^2}{\partial t^2}\mathbf{u} - \nabla \cdot \tau = \mathbf{F}(x, t)$$

Mass conservation

$$\frac{\partial}{\partial t}\eta - \nabla \cdot \mathbf{q} = G(x, t)$$

ρ density

Mathematical Model

The P.D.E.

Fully dynamic poroelasticity

$$\rho \frac{\partial^2}{\partial t^2}\mathbf{u} - \lambda^* \nabla \left(\frac{\partial}{\partial t} \nabla \cdot \mathbf{u} \right) - (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) - \mu \nabla \cdot (\nabla \mathbf{u}) + \alpha \nabla p = \mathbf{F}(x, t)$$

$$\frac{\partial}{\partial t}(c_0 p + \alpha \nabla \cdot \mathbf{u}) - \nabla \cdot (\kappa \nabla p) = G(x, t)$$

Quasistatic poroelasticity

$$-(\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) - \mu \nabla \cdot (\nabla \mathbf{u}) + \alpha \nabla p = \mathbf{F}(x, t)$$

$$\frac{\partial}{\partial t}(c_0 p + \alpha \nabla \cdot \mathbf{u}) - \nabla \cdot (\kappa \nabla p) = G(x, t)$$

In $\Omega \times (0, T)$

Mathematical Model

Boundary Conditions

$$\mathbf{u} = \mathbf{u}_c \quad \text{on} \quad \Gamma_c \times (0, T)$$

$$\Big[(\lambda + \mu)\nabla \cdot \mathbf{u} I + \mu \nabla \mathbf{u}\Big]\mathbf{n} - \beta \alpha p \mathbf{n} \chi_{tf} = \mathbf{g} \quad \text{on} \quad \Gamma_t \times (0, T)$$

$$p = p_d \quad \text{on} \quad \Gamma_d \times (0, T)$$

$$-\frac{\partial}{\partial t}\Big((1-\beta)\alpha \mathbf{u} \cdot \mathbf{n}\Big)\chi_{tf} + \kappa \nabla p \cdot \mathbf{n} = j \quad \text{on} \quad \Gamma_f \times (0, T)$$

$\Gamma = \overline{\Gamma}_c \cup \overline{\Gamma}_t$, $\Gamma_c \cap \Gamma_t = \emptyset$, also $\Gamma = \overline{\Gamma}_d \cup \overline{\Gamma}_f$, $\Gamma_d \cap \Gamma_f = \emptyset$, and $\chi_{tf} = \chi_{\Gamma_t \cap \Gamma_f}$

Mathematical Model

Initial Conditions

$$c_0 p + \alpha \nabla \cdot \mathbf{u} = v_0 \quad \text{on} \quad \Omega \quad \text{at} \quad t = 0$$

$$(1 - \beta)\alpha \mathbf{u} \cdot \mathbf{n} = v_1 \quad \text{on} \quad \Gamma_{tf} \quad \text{at} \quad t = 0$$

Mixed Formulation

Quasistatic poroelasticity

$$-(\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) - \mu\nabla \cdot (\nabla \mathbf{u}) + \alpha\nabla p = \mathbf{F}(x, t)$$

$$\frac{\partial}{\partial t}(c_0 p + \alpha\nabla \cdot \mathbf{u}) - \nabla \cdot (\mathbf{z}) = G(x, t)$$

$$\kappa^{-1}\mathbf{z} - \nabla p = 0$$

In $\Omega \times (0, T)$