A NOTE ON MULTIPLICATION AND COMPOSITION OPERATORS IN LORENTZ SPACES

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Abstract. In this paper, we revisit the Lorentz spaces $L(p,q)$ for $p>1, q>0$ defined by G. G. Lorentz in the nineteen fifties and we show how the atomic decomposition of the spaces $L(p,1)$ obtained by De Souza in 2010 can be used to characterize the multiplication and composition operators on these spaces. These characterizations, though obtained from a completely different perspective, confirm the various results obtained by S. C. Arora, G. Datt and S. Verma in different variants of the Lorentz Spaces.

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1. Introduction

In the early 1950s, G. G. Lorentz introduced the now famous Lorentz Spaces $L(p,q)$ in his papers [6] and [7] as a generalization of the L^p spaces. The parameters p and q encode the information about the size of a function, that is, how tall and how spread out a function is. The Lorentz spaces are quasi-Banach spaces in general but the Lorentz quasi-norm of a function has better control over the size of the function than the L^p norm, via the parameters p and q, making the spaces very useful. We are mostly concerned with studying the multiplication and composition operators on Lorentz spaces. This has been studied before by various authors in particular by S. C. Arora, G. Datt and S. Verma in [2],[3],[4] and [5]. In this paper, the results we obtain are in accordance with what these authors have found before. We believe that the techniques and relative simplicity of our approach are worth reporting to further enrich the topic. Our results, found on the boundary of the unit disc due to the original focus by De Souza in [1], will show how one can use the atomic characterization of the Lorentz space $L(p,1)$ in the study of multiplication and composition operators in the spaces $L(p,q)$.

2. Preliminaries

Let (X,μ) be a measure space.

Definition 2.1. Let f be a complex-valued function defined on X. The *decreasing rearrangement* of f is the function f^* defined on $[0,\infty)$ by

$$f^*(t)=\inf\{y>0 : d(f,y)\le t\}\ ,$$

where $d(f,y)=\mu(\{x : |f(x)|>y\})$ is the *distribution* of the function f.

Definition 2.2. Given a measurable function f on (X,μ) and $0<p,q\le\infty$, define

$$\|f\|_{L(p,q)} = \begin{cases} \left(\dfrac{q}{p}\displaystyle\int_0^\infty \left(t^{\frac{1}{p}} f^*(t)\right)^q \dfrac{dt}{t}\right)^{\frac{1}{q}} & \text{if } q<\infty, \\ \sup\limits_{t>0} t^{\frac{1}{p}} f^*(t) & \text{if } q=\infty. \end{cases}$$

The set of all functions f with $\|f\|_{L(p,q)}<\infty$ is called *the Lorentz Space* with indices p and q and denoted by $L(p,q)(X,\mu)$.

We now consider the measure μ on X to be finite. Let $g: X\to X$ be a $\mu-$measurable function such that $\mu(g^{-1}(A))\le C\mu(A)$ for a $\mu-$measurable set $A\subseteq[0,2\pi]$ and for an absolute constant C. Here $g^{-1}(A)$ refers to the pre-image of the set A.

Remark 2.3. It is important to note that $\|g\| = \sup\limits_{\mu(A)\neq 0}\dfrac{\mu(g^{-1}(A))}{\mu(A)}$ is not a necessarily a norm.

Definition 2.4. For a given function g, we define the multiplication operator T_g on Lorentz spaces as $T_g(f)=f\cdot g$ and the composition operator C_g as $C_g(f)=f\circ g$.

The following two results are used in our proofs. The first is a result of De Souza [1] which gives an analytic characterization of $L(p,1)$. The second is the Marcinkiewicz Interpolation Theorem (see [8]) which we state for completeness of presentation.

Theorem 2.5 (De Souza [1]). *A function $f\in L(p,1)$ for $p>1$ if and only if $f(t)=\sum_{n=1}^\infty c_n\chi_{A_n}(t)$ with $\sum_{n=1}^\infty |c_n|\mu^{\frac{1}{p}}(A_n)<\infty$, where is μ is measure on X and A_n are μ-measurable sets in X. Moreover, $\|f\|_{L(p,1)}\cong \inf\sum_{n=1}^\infty |c_n|\mu^{\frac{1}{p}}(A_n)$, where the infimum is taken over all possible representations of f.*

Theorem 2.6 (Marcinkiewicz). *Assume that for $0<p_0\neq p_1\le\infty$, for all $q>0$, for all measurable subsets A of X, there are some constants $0<M_0,M_1<\infty$ such that for a linear or quasi-linear operator T_g*

a) $\|T_g\chi_A\|_{L(p_0,\infty)}\le M_0\mu^{\frac{1}{p_0}}(A)$
b) $\|T_g\chi_A\|_{L(p_1,\infty)}\le M_1\mu^{\frac{1}{p_1}}(A)$.

Then there is some $M>0$ such that $\|T_gf\|_{L(p,q)}\le M\|f\|_{L(p,q)}$ for $\frac{1}{p}=\frac{\theta}{p_0}+\frac{1-\theta}{p_1}$, $\quad 0<\theta<1$.

One implication of Theorem 2.5 is that it can be used to prove and justify a theorem of Stein and Weiss [9]. That is, to show that linear operators $T: L(p,1)\to B$ are bounded, where B is Banach space closed under absolute value and satisfying $\|f\|_B=\||f|\|_B$, all one needs to show is that $\|T\chi_A\|_B\le M\mu^{\frac{1}{p}}(A), \quad p>1$. Theorem 2.6 will be used to show that results valid on $L(p,1)$ are also valid on $L(p,q)$.

Definition 2.7. We denote by M_r^p the set of real-valued functions defined on $X=[0,2\pi]$ such that

$$\|f\|_{M_r^p} = \sup_{x>0}\left(\frac{r}{px^{1/p}}\int_0^x \left(f^*(t)t^{1/p}\right)^r\frac{dt}{t}\right)^{1/r}<\infty, \tag{1}$$

where $1\le p\le r<\infty$.

We will show that the space M_r^p is equivalent to a weak L^k space for some k that depends on p and r and $\|\cdot\|_{M_r^p}$ is quasi-norm.

Lemma 2.8. *$\|\cdot\|_{M_r^p}$ a quasi-norm on M_r^p.*

Proof. $f^* \geq 0$ by definition. This implies that $||f||_{M_r^p} \geq 0$. Moreover, $||f||_{M_r^p} = 0$ implies that for all $0 < x \leq 2\pi$, $\int_0^x \left(f^*(t)t^{\frac{1}{p}}\right)^r \frac{dt}{t} = 0$. Hence we have $f^* = 0 \quad \mu$-a.e, thus $f = 0$ since f is a representative of an equivalence class. Now let $k \neq 0$ be a real constant, $f \in M_r^p$ and $x \in (0, 2\pi]$. Noting $(kf)^* = |k|f^*$, the homogeneity condition $\|kf\|_{M_r^p} = |k|\|f\|_{M_r^p}$ follows trivially. Let $f, g \in M_r^p$. Since $(f+g)^*(t) \leq f^*(t/2) + g^*(t/2)$, for any $x \in (0, 2\pi]$, we have

$$\begin{aligned}\int_0^x \left((f+g)^*(t)t^{\frac{1}{p}}\right)^r \frac{dt}{t} &\leq 2^{r-1}\left(\int_0^x \left(f^*(t/2)t^{\frac{1}{p}}\right)^r \frac{dt}{t} + \int_0^x \left(g^*(t/2)t^{\frac{1}{p}}\right)^r \frac{dt}{t}\right) \\ &\leq 2^{\frac{r}{p}+r-1}\left(\int_0^{\frac{1}{2}x} \left(f^*(t)t^{\frac{1}{p}}\right)^r \frac{dt}{t} + \int_0^{\frac{1}{2}x} \left(g^*(t)t^{\frac{1}{p}}\right)^r \frac{dt}{t}\right)\end{aligned}$$

Since $(a+b)^{1/r} \leq a^{1/r} + b^{1/r}$ for $a, b > 0$, we have

$$||f+g||_{M_r^p} \leq 2^{\frac{1}{p}-\frac{1}{r}+1}(\|f\|_{M_r^p} + \|g\|_{M_r^p}), \text{ with } 2^{\frac{1}{p}-\frac{1}{r}+1} > 1 \text{ for } r, p > 1.$$

□

Theorem 2.9. $M_r^p \cong L(pr', \infty)$ *where* $r, r' \geq 1, \frac{1}{r} + \frac{1}{r'} = 1$

Proof. Suppose $g \in M_r^p$. There is an absolute constant C such that for all $x > 0$,

$$C \geq \left(\frac{r}{px^{1/p}}\int_0^x \left(g^*(t)t^{1/p}\right)^r \frac{dt}{t}\right)^{1/r} \geq \left(\frac{r}{px^{1/p}}(g^*(x))^r \int_0^x t^{\frac{r}{p}-1}dt\right)^{1/r} = x^{\frac{1}{pr'}}g^*(x).$$

Thus $\sup_{x>0} x^{\frac{1}{pr'}}g^*(x) \leq C$ implying that $g \in L(pr', \infty)$.

Conversely, let $g \in L(pr', \infty)$. Then there is an absolute constant C such that, $g^*(t) \leq Ct^{-1/pr'}$. This implies that $(g*(t)t^{1/p})^r \leq C^r t^{1/p}$. Thus

$$\sup_{x>0}\left(\frac{r}{px^{1/p}}\int_0^x \left(g^*(t)t^{1/p}\right)^r \frac{dt}{t}\right)^{1/r} \leq \sup_{x>0}\left(C^r \frac{r}{px^{1/p}}\int_0^x t^{1/p}\frac{dt}{t}\right)^{1/r} = Cr^{1/r}$$

This implies that $g \in M_r^p$. □

Remark 2.10. One can easily see from Theorem 2.9 that $M_\infty^p \cong L(p, \infty)$ and $M_1^p \cong L^\infty$. Moreover, $\|g\|_{M_1^p} = \|g\|_\infty$. To see this, note that

$$\|g\|_{M_1^p} = \sup_{x>0}\left(\frac{1}{px^{1/p}}\int_0^x g^*(t)t^{1/p}\frac{dt}{t}\right) \leq \|g\|_\infty \sup_{x>0}\left(\frac{1}{px^{1/p}}\int_0^x t^{1/p-1}dt\right) = \|g\|_\infty$$

and

$$\|g\|_{M_1^p} \geq \frac{g^*(x)}{px^{1/p}}\int_0^x t^{1/p-1}dt = g^*(x)$$

for all x since g^* is decreasing. Taking the limit as $x \to 0$, we see that $\|g\|_{M_1^p} \geq g^*(0) = \|g\|_\infty$.

3. Main Results

3.1. Multiplication Operators.

Theorem 3.1 (Multiplication Operator on $L(p,1)$)**.** *The multiplication operator $T_g : L(p,1) \to L(p',1)$ for $p' \geq p > 1$ is bounded if and only $g \in L^\infty$. Moreover, $\|Tg\| = \|g\|_\infty$.*

Proof. It is convenient to use M_1^p which is equivalent to L^∞. Assume that $\|T_g f\|_{L(p',1)} \leq C\|f\|_{L(p,1)}$. Then for $f = \chi_{[0,x]}$ where $x \in (0, 2\pi]$,

$$\int_0^{2\pi} (T_g\chi_{[0,x]})^*(t)t^{\frac{1}{p'}-1}dt = \int_0^x g^*(t)t^{\frac{1}{p'}-1}dt \leq C\int_0^{2\pi} \chi_{[0,x]}^*(t)t^{\frac{1}{p}-1}dt = Cpx^{\frac{1}{p}}.$$

Multiplying and dividing the integrand on the left by $t^{\frac{1}{p}-1}$, we get

$$\int_0^x g^*(t)t^{\frac{1}{p}-1}t^{\frac{p-p'}{pp'}}dt \leq Cpx^{\frac{1}{p}}.$$

Since $t \mapsto t^{\frac{p-p'}{pp'}}$ is decreasing on $[0,x]$ and $0 < x \leq 2\pi$, we have

$$\frac{1}{px^{1/p}}\int_0^x g^*(t)t^{\frac{1}{p}-1}dt \leq C(2\pi)^{\frac{p-p'}{pp'}}$$

Taking the supremum over all $x > 0$, we have that $g \in M_1^p$.

Assume that $g \in M_1^p$ and $x > 0$. Since $p' > p$ we have

$$\|T_g\chi_{[0,x]}\|_{L(p',1)} = \int_0^x g^*(t)t^{\frac{1}{p'}-1}dt \leq \int_0^x g^*(t)t^{\frac{1}{p}-1}dt.$$

And so,

$$\|T_g\chi_{[0,x]}\|_{L(p',1)} \leq M\|\chi_{[0,x]}\|_{L(p,1)} \quad \text{where } M = \sup_{x>0}\frac{1}{px^{1/p}}\int_0^x g^*(t)t^{\frac{1}{p}-1}dt\,. \tag{2}$$

Using the atomic decomposition of $L(p,1)$, we get

$$\|T_g f\|_{L(p',1)} \leq M'\|f\|_{L(p,1)} \quad \text{for some positive constant } M'.$$

To prove the second part of the theorem, first note that the expression in (2) gives that $\|Tg\| \leq \|g\|_\infty$. Now take $f = \frac{1}{x^{1/p}}\chi_{[0,x]}$. We can easily see that $\|f\|_{L(p,1)} = 1$ and $\|T_g f\|_{L(p',1)} \geq g^*(x)$ for $x \in [0,2\pi]$ since g^* is decreasing. Now taking the sup over $\|f\|_{L(p,1)} \leq 1$ and the limit as $x \to 0$ gives $\|T_g\| \geq \|g\|_\infty$. Thus $\|T_g\| = \|g\|_\infty$. □

The following theorem, which is equivalent to Theorem 1.1 of [5], follows from Theorems 2.6 and 3.1.

Theorem 3.2 (Multiplication Operator on $L(p,q)$)**.** *The multiplication operator $T_g : L(p,q) \to L(p,q)$ is bounded if and only if $g \in L^\infty$ for $1 < p \leq \infty,\ 1 < q \leq \infty$. Moreover, $\|T_g\| = \|g\|_\infty$.*

Remark 3.3. Since, by Theorem 2.9, $M_1^p \subseteq M_r^p$ for $r > 1$, the theorem implies that if the multiplication operator $T_g : L(p,q) \to L(p,q)$ defined by $T_g f = g \cdot f$ is bounded, then $g \in M_r^p$ for $p, q > 1$.

Remark 3.4. Note that the previous theorem can be proved using the norm

$$\|f\|_{L(p,1)} = \sup_{\substack{A\subseteq X\\ \mu(A)\neq 0}} \frac{1}{\mu^{\frac{1}{p}}(A)} \int_0^{\mu(A)} g^*(t)t^{\frac{1}{p}-1}dt$$

on $L(p,1)$. In fact this norm is the motivation for the definition of the spaces M_r^p.

We show in the next result in what Lorentz space to expect the product of two functions from different Lorentz spaces.

Theorem 3.5. *If $f \in L(p_1,q_1)$ and $g \in L(p_2,q_2)$ where $1 < p_1,p_2,q_1,q_2 < \infty$, then $g\cdot f \in L(r,s)$ where $\dfrac{1}{r} = \dfrac{1}{p_1}+\dfrac{1}{p_2}$, $\dfrac{1}{s} = \dfrac{1}{q_1}+\dfrac{1}{q_2}$.*

Proof. Given $1 < p_1,p_2,q_1,q_2 < \infty$, assume $f \in L(p_1,q_1)$ and $g \in L(p_2,q_2)$. Let r,s be such that $1/r = 1/p_1+1/p_2$ and $1/s = 1/q_1+1/q_2$. Since $(f\cdot g)^*(t) \leq f^*(t)g^*(t)$, we have

$$\int_0^{2\pi} \left((f\cdot g)^*(t)t^{\frac{1}{r}}\right)^s \frac{dt}{t} \leq \int_0^{2\pi} \left(f^*(t)t^{\frac{1}{p_1}}\right)^s \cdot \left(g^*(t)t^{\frac{1}{p_2}}\right)^s \frac{dt}{t}$$

Using Holder's inequality on the RHS with $s/q_1 + s/q_2 = 1$ we have

$$\int_0^{2\pi} \left((f\cdot g)^*(t)t^{\frac{1}{r}}\right)^s \frac{dt}{t} \leq \left(\int_0^{2\pi} \left(f^*(t)t^{\frac{1}{p_1}}\right)^{q_1} \frac{dt}{t}\right)^{\frac{s}{q_1}} \cdot \left(\int_0^{2\pi} \left(g^*(t)t^{\frac{1}{q_2}}\right)^{q_2} \frac{dt}{t}\right)^{\frac{s}{q_2}}$$

Thus, we have

$$\|g\cdot f\|_{L(r,s)} \leq \|f\|_{L(p_1,q_1)} \cdot \|g\|_{L(p_2,q_2)}\,.$$

□

Theorem 3.6. *If $g \in M_r^p$, then $T_g : L(q,s) \to L\left(\frac{pqr'}{pr'+q}, s\right)$ is bounded, where $\frac{1}{r}+\frac{1}{r'} = 1$ and for $s > 0$ and $p,q > 1$.*

Proof. Let $g \in M_r^p \cong L(pr',\infty)$.

$$\begin{aligned} \|T_g f\|_{L(k,s)}^s &\leq \int_0^{2\pi} \left(g^*(t)\cdot f^*(t)t^{1/k}\right)^s \frac{dt}{t} \\ &= \int_0^{2\pi} \left(g^*(t)t^{1/pr'}\cdot f^*(t)t^{1/q}\right)^s \frac{dt}{t} \quad \text{if } \frac{1}{k} = \frac{1}{q}+\frac{1}{qr'}. \end{aligned}$$

Therefore

$$\|T_g f\|_{L(k,s)}^s \leq \sup_{t>0} \left(g^*(t)t^{1/pr'}\right)^s \cdot \int_0^{2\pi} \left(f^*(t)t^{1/q}\right)^s \frac{dt}{t}.$$

That is,

$$\|T_g f\|_{L(k,s)} \leq \|g\|_{M_r^p} \cdot \|f\|_{L(q,s)}\,, \quad \text{where } k = \frac{pr'q}{pr'+q}.$$

□

Noting that $M_r^p \cong L(pr',\infty)$, $r' = r/(r-1)$, it is easy to see that Theorem 3.6 shows that the result of Theorem 3.5 extends to the case where $q_2 = \infty$.

3.2. **Composition Operators.**

Theorem 3.7. *The composition operator $C_g : L(p,q) \to L(p,q)$ is bounded if and only if there is an absolute constant C such that*

$$\mu(g^{-1}(A)) \leq C\mu(A) \tag{3}$$

for all $\mu-$measurable sets $A \subseteq [0,2\pi]$ and for $1 < p \leq \infty, 1 \leq q \leq \infty$. Moreover, $\|C_g\| = \|g\|^{1/p}$.

Proof. We will prove this theorem for $L(p,1)$ and use the Interpolation Theorem to conclude for $L(p,q)$.

First assume that $C_g : L(p,1) \to L(p,1)$ is bounded, that is, there is an absolute constant C such that

$$\|C_g f\|_{L(p,1)} \leq C\|f\|_{L(p,1)}. \tag{4}$$

Let A be a $\mu-$measurable set in $[0,2\pi]$ and let $f = \chi_A$. Then (4) is equivalent to

$$\|C_g\chi_A\|_{L(p,1)} \leq C\|\chi_A\|_{L(p,1)} \Leftrightarrow \frac{1}{p}\int_0^{2\pi} \big(C_g\chi_A\big)^*(t)t^{\frac{1}{p}-1}dt \leq \frac{C}{p}\int_0^{2\pi} \chi_A^*(t)t^{\frac{1}{p}-1}dt$$

that is,

$$\frac{1}{p}\int_0^{2\pi} (\chi_A \circ g)^*(t)t^{\frac{1}{p}-1}dt \leq \frac{C}{p}\int_0^{2\pi} \chi_{[0,\mu(A)]}(t)t^{\frac{1}{p}-1}dt$$

Since $(\chi_A \circ g) = \chi_{g^{-1}(A)}$, then $(\chi_A \circ g)^* = \chi_{[0,\mu(g^{-1}(A))]}$. Therefore the previous inequality gives

$$\frac{1}{p}\int_0^{\mu(g^{-1}(A))} t^{\frac{1}{p}-1}dt \leq \frac{C}{p}\int_0^{\mu(A)} t^{\frac{1}{p}-1}dt.$$

And hence

$$\mu(g^{-1}(A)) \leq C^p\mu(A).$$

On the other hand, assume that there is some constant $C > 0$ such that $\mu(g^{-1}(A)) \leq C\mu(A)$. Then

$$\begin{aligned} \|C_g\chi_A\|_{L(p,1)} = \frac{1}{p}\int_0^{2\pi} \big(\chi_A \circ g\big)^*(t)t^{\frac{1}{p}-1}dt &= \frac{1}{p}\int_0^{2\pi} \chi_{[0,\mu(g^{-1}(A))]}(t)t^{\frac{1}{p}-1}dt \\ &= \big(\mu(g^{-1}(A))\big)^{\frac{1}{p}} \leq C^{\frac{1}{p}}\big(\mu(A)\big)^{\frac{1}{p}}. \end{aligned}$$

Consequently,

$$\|C_g\chi_A\|_{L(p,1)} \leq C^{\frac{1}{p}}\big(\mu(A)\big)^{\frac{1}{p}}.$$

As a consequence of (2.5) or the result by Weiss and Stein in [9], we have

$$\|C_g f\|_{L(p,1)} \leq C^{\frac{1}{p}}\|f\|_{L(p,1)}.$$

To prove the second part of the theorem, note that from the above, we have

$$\|C_g\| = \sup_{\|f\|_{L(p,1)}\leq 1} \frac{\|C_g f\|_{L(p,1)}}{\|f\|_{L(p,1)}} \leq C^{1/p}.$$

But $\inf\{C : \mu(g^{-1}(A)) \leq C\mu(A)\} = \|g\|$. Thus $\|C_g\| \leq \|g\|^{1/p}$. To obtain the other inequality, let $f = \frac{1}{[\mu(A)]^{1/p}}\chi_A$. This gives $\|f\|_{L(p,1)} = 1$ and

$$\|C_g f\|_{L(p,1)} = \left\{\frac{\mu(g^{-1}(A))}{\mu(A)}\right\}^{1/p}$$

Thus

$$\|C_g\| = \sup_{\|f\|_{L(p,1)}\leq 1} \|C_g f\|_{L(p,1)} \geq \sup_{\mu(A)\neq 0} \left\{ \frac{\mu(g^{-1}(A))}{\mu(A)} \right\}^{1/p} = \|g\|^{1/p} .$$

Now to show the result for $L(p,q)$, note that the operator C_g is linear on $L(p,q)$ and that for p_0 and p_1 such that $p_0 < p < p_1$, we have $\|C_g \chi_A\|_{L(p_0,1)} \leq M_0 \big(\mu(A)\big)^{\frac{1}{p_0}}$ and $\|C_g \chi_A\|_{L(p_1,1)} \leq M_1 \big(\mu(A)\big)^{\frac{1}{p_1}}$. Since $L(p_i,1) \subseteq L(p_i,\infty), i = 0,1$, then for some absolute constants C_0 and C_1 we have

$$\|C_g \chi_A\|_{L(p_0,\infty)} \leq C_0 \big(\mu(A)\big)^{\frac{1}{p_0}} .$$

and

$$\|C_g \chi_A\|_{L(p_1,\infty)} \leq C_0 \big(\mu(A)\big)^{\frac{1}{p_1}} .$$

Hence by the Interpolation Theorem we conclude that there is a constant $C > 0$ such that

$$\|C_g f\|_{L(p,q)} \leq C \|f\|_{L(p,q)} \quad \text{for } p_0 < p < p_1, \text{ for all } q \text{ and for all } f \in L(p,q).$$

□

Remark 3.8. The necessary and sufficient condition (3) makes intuitive sense if we consider a variety of measures. Let us consider two of them.

(1) If μ is the Lebesgue measure and X happens to be an interval, then it suffices to take g as the left multiplication by an absolute constant a to achieve (3).
(2) If instead μ is the Haar measure, by taking $X = (0,\infty)$, the locally compact topological group of nonzero real numbers with multiplication as operation, then for any Borel set $A \subseteq X$, we have $\mu(A) = \int_A |t|^{-1} dt$. Hence (3) is achieved for a measurable function g such that $g^{-1}(A) \subseteq A$. The left multiplication by the reciprocal of an absolute constant a would be enough.

Remark 3.9. The results in Theorems 3.6 and 3.7 are in accordance with the results of S. C. Arora, G. Datt and S. Verma in [4] and [5]. In fact, even though they obtained their results in a more general version of Lorentz spaces, their necessary and sufficient conditions for boundedness of the multiplication and composition operators are the same as ours.

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