Optimal Control for a Differential Susceptibility and Infectivity HIV/AIDS Model

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Outline

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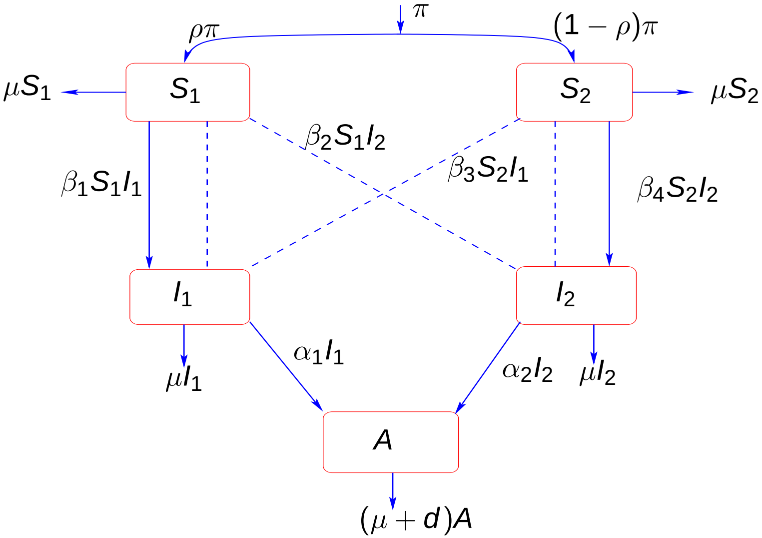
A diagrammatic representation of HIV model without AIDs interaction



HIV/AIDS: Low Risk and High Risk Population Model

$$
\begin{aligned}
\frac{dS_1}{dt} &= \rho\Pi - \mu S_1 - \beta_1 S_1 I_1 - \beta_2 S_1 I_2 \\
\frac{dI_1}{dt} &= \beta_1 S_1 I_1 + \beta_2 S_1 I_2 - (\mu + \alpha_1) I_1 \\
\frac{dS_2}{dt} &= (1-\rho)\Pi - \mu S_2 - \beta_3 S_2 I_1 - \beta_4 S_2 I_2 \\
\frac{dI_2}{dt} &= \beta_3 S_2 I_1 + \beta_4 S_2 I_2 - (\mu + \alpha_2) I_2 \\
\frac{dA}{dt} &= \alpha_1 I_1 + \alpha_2 I_2 - (\mu + d) A
\end{aligned}
$$

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Positivity and Boundedness

Theorem: Given $S_1(0) \geq 0, S_2(0) \geq 0, I_1(0) \geq 0, I_2 \geq 0, A \geq 0$, then the solutions: $(S_1(t), S_2(t), I_1(t), I_2(t), A(t))$ of the model are positively invariant for al $t > 0$.

Theorem: All solutions $(S_1(t), S_2(t), I_1(t), I_2(t), A(t))$ of the model are bounded.

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Disease Free and Endemic Equilibria

$$DFE = \left(S_1^0, S_2^0, I_1^0, I_2^0, A^0\right) = \left(\frac{\rho\pi}{\mu}, \frac{(1-\rho)\pi}{\mu}, 0, 0, 0\right)$$

Endemic Equilibrium

$$\begin{aligned}
S_1^* &= \frac{\rho\pi}{\mu + \beta_1 I_1^* + \beta_2 I_2^*} \\
S_2^* &= \frac{(1-\rho)\pi}{\mu + \beta_3 I_1^* + \beta_4 I_2^*} \\
I_1^* &= \frac{\beta_2 S_1 I_2^*}{\mu + \alpha_1 - \beta_1 S_1^*} \\
I_2^* &= \frac{\beta_3 S_2 I_1^*}{\mu + \alpha_2 - \beta_4 S_2^*} \\
A^* &= \frac{\alpha_1 I_1^* + \alpha_2 I_2^*}{\mu + d}
\end{aligned}$$

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Basic Reproduction Number and Stability of DFE

Theorem: If $R_0 < 1$, then the disease-free equilibrium point is locally asymptotically stable and unstable for $R_0 > 1$.

The holistic approach-induced reproduction number is

$$R_0 = \frac{\mu\pi(\mu+\alpha_1)(1-\rho)\beta_4 + (\mu+\alpha_2)\mu\rho\pi\beta_1 + \rho(1-\rho)\pi^2\beta_2\beta_3}{\mu^2(\mu+\alpha_1)(\mu+\alpha_2) + \rho(1-\rho)\pi^2\beta_1\beta_4} < 1$$

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Introduction

- Interested in intervention strategies for the spread of HIV and AIDS through the population.
- Two potential options:
  1. Use of educational programs within the population
  2. Provision of treatment drugs to infected classes
- For this study, we started by modeling the implementation of educational campaigns
  - The role of education control is to move people from the high risk susceptibles and infectives to the low-risk susceptible and infectives
  - Controls were chosen to be nonlinear
- Used methods in optimal control theory to help develop an optimal strategy for implementing an education campaign in the population

Model with Education Campaigns

$$\frac{dS_1}{dt} = \rho\Pi - \mu S_1 - \beta_1 S_1 I_1 - \beta_2 S_1 I_2 + \gamma_S S_2 \quad (1)$$

$$\frac{dI_1}{dt} = \beta_1 S_1 I_1 + \beta_2 S_1 I_2 - (\mu + \alpha_1) I_1 + \gamma_I I_2 \quad (2)$$

$$\frac{dS_2}{dt} = (1-\rho)\Pi - \mu S_2 - \beta_3 S_2 I_1 - \beta_4 S_2 I_2 - \gamma_S S_2 \quad (3)$$

$$\frac{dI_2}{dt} = \beta_3 S_2 I_1 + \beta_4 S_2 I_2 - (\mu + \alpha_2) I_2 - \gamma_I I_2 \quad (4)$$

$$\frac{dA}{dt} = \alpha_1 I_1 + \alpha_2 I_2 - (\mu + d) A \quad (5)$$

γ_S is the rate of progression from high risk susceptibles to low risk susceptibles

γ_I is the rate of progression from high risk infectives to low risk infectives

Objective Functional

Want to maximize to following:

$$J(S_1, S_2, I_1, I_2, \gamma_S, \gamma_I, t) = \int_0^T (S_1 - d(I_1 + I_2) - c((\gamma_I)^2 + (\gamma_S)^2))dt$$

subject to the constraints: $(1) - (5)$ and

$$0 \leq \gamma_S \leq \gamma_S^{max} \qquad 0 \leq \gamma_I \leq \gamma_I^{max}$$

and initial conditions:

$$(S_1(0), S_2(0), I_1(0), I_2(0), A(0)) = (S_{01}, S_{02}, I_{01}, I_{02}, A_0)$$

Pontryagin's Maximum Principle

- The principle converts the problem of maximizing the objective functional subject to the constraints and initial conditions to a problem of maximizing the Hamiltonian with respect to the controls

Forming the Hamiltonian:

$$
\begin{aligned}
H &= S_1 - d(I_1 + I_2) - c((\gamma_I)^2 + (\gamma_S)^2)) \\
&+ \lambda_1(\rho\Pi - \mu S_1 - \beta_1 S_1 I_1 - \beta_2 S_1 I_2 + \gamma_S S_2) \\
&+ \lambda_2(\beta_1 S_1 I_1 + \beta_2 S_1 I_2 - (\mu + \alpha_1) I_1 + \gamma_I I_2) \\
&+ \lambda_3((1-\rho)\Pi - \mu S_2 - \beta_3 S_2 I_1 - \beta_4 S_2 I_2 - \gamma_S S_2) \\
&+ \lambda_4(\beta_3 S_2 I_1 + \beta_4 S_2 I_2 - (\mu + \alpha_2) I_2 - \gamma_I I_2) \\
&+ \lambda_5(\alpha_1 I_1 + \alpha_2 I_2 - (\mu + d) A)
\end{aligned}
$$

Adjoint Equations

$$\begin{aligned}
\lambda_1' = -\frac{\partial H}{\partial S_1} &= -[1 - (\mu + \beta_1 I_1 + \beta_2 I_2)\lambda_1 + (\beta_1 I_1 + \beta_2 I_2)\lambda_2] \\
\lambda_2' = -\frac{\partial H}{\partial I_1} &= -[-d - \beta_1 S_1 \lambda_1 + (\beta_1 S_1 - (\mu + \alpha_1))\lambda_2 \\
&- \beta_3 S_2 \lambda_3 + \beta_3 S_2 \lambda_4 + \alpha_1 \lambda_5] \\
\lambda_3' = -\frac{\partial H}{\partial S_2} &= -[\gamma_S \lambda_1 - (\mu + \beta_3 I_1 + \beta_4 I_2 + \gamma_S)\lambda_3 + (\beta_3 I_1 + \beta_4 I_2)\lambda_4] \\
\lambda_4' = -\frac{\partial H}{\partial I_2} &= -[-d - \beta_2 S_1 \lambda_1 + \gamma_I \lambda_2 \\
&- \beta_4 S_2 \lambda_3 + (\beta_4 S_2 - (\mu + \alpha_2) - \gamma_I)\lambda_4 + \alpha_2 \lambda_5] \\
\lambda_5' = -\frac{\partial H}{\partial A} &= \lambda_5(\mu + d)
\end{aligned}$$

and transversality conditions:

$$\lambda_1(T) = \lambda_2(T) = \lambda_3(T) = \lambda_4(T) = \lambda_5(T) = 0$$

Optimality Condition

- Since controls are nonlinear in the control, can solve directly:

$$\begin{aligned} 0 &= \frac{dH}{d\gamma_S} = -2c\gamma_S + \lambda_1 S_2 - \lambda_3 S_2 \\ \Rightarrow \quad \gamma_S^* &= \frac{(\lambda_1 - \lambda_3)S_2}{2c} \end{aligned}$$

and

$$\begin{aligned} 0 &= \frac{dH}{d\gamma_I} = -2c\gamma_I + \lambda_2 I_2 - \lambda_4 I_2 \\ \Rightarrow \quad \gamma_I^* &= \frac{(\lambda_2 - \lambda_4)I_2}{2c} \end{aligned}$$

Optimality Condition

- Using the bounds on the controls, can characterize the optimal controls as:

$$\gamma_S^* = max\{min\{\gamma_S^{max}, \frac{(\lambda_1 - \lambda_3)S_2}{2c}\}, 0\}$$
$$\gamma_I^* = max\{min\{\gamma_I^{max}, \frac{(\lambda_2 - \lambda_4)I_2}{2c}\}, 0\}$$

- Next step is find numerical approximations for the corresponding states, adjoints, and the optimal intervention strategy.

Outline

Future Work

- Global Stability of the Disease Free Equilibrium
- Stability of the Endemic Equilibrium
- Simulations of the Basic Model
- Numerical Approximations
- Alternative Intervention Strategies

Thank you

Asante

Kealeboga