MINI-COURSE ON FUNCTIONAL ANALYSIS

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Chapter 1

Vector Spaces

Definition 1.1. *A vector space consists of a non-empty set X where its elements are called vectors and a scalar field F in which its elements are called scalars, endowed with two operations one called addition and the other called multiplication that satisfy several properties.*

Remark 1.1. *Even though the scalar field F can be very general, in this course we just consider F as being the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$.*

Addition operation:	Scalar operation:
$+$: $\mathrm{X} \times X \longrightarrow F$	$\cdot : F \times X \longrightarrow X$
$(x, y) \longmapsto x + y$	$(\alpha, x) \longmapsto \alpha \cdot x$

These operations must satisfy the following properties:

1. $(x + y) + z = x + (y + z), \quad \forall x, y, z \in X$
2. $x + y = y + x, \quad \forall x, y \in X$
3. $\exists 0 \in X$ such that $x + 0 = 0 + x = x, \quad \forall x \in X$
4. Given $x \in X, \exists x' \in X$ such that $x + x' = x' + x = 0$
5. $\lambda \cdot (x + y) = \lambda \cdot x + \lambda \cdot y, \quad \forall \lambda \in F, \quad \forall x, y \in X$
6. $(\lambda + \mu) \cdot x = \lambda \cdot x + \mu \cdot x, \quad \forall \lambda, \mu \in F, \quad x \in X$
7. $\lambda(\mu) \cdot x = (\lambda\mu) \cdot x, \quad \forall \lambda, \mu \in F, \forall x \in X$
8. There is an element $1 \in X$ such that $1 \cdot x = x, \quad \forall x \in X$

Remark 1.2. *The set X endowed with these two operation and satisfying these eight properties is called a **vector space**, and denoted by $(X, +, \cdot)$.*

Remark 1.3. *The set X endowed with the operation addition and satisfying the first four propoerties is called an **additive group** and is denoted by $(X, +)$.*

Example 1.1. Take $X = \mathbb{R}$ the real numbers, endowed with the usual operations of addition and multiplication. Then $(\mathbb{R}, +, .)$ is the vector space of real numbers.

Example 1.2. Let $X = \mathbb{R}^n = \{x = (x_1, \cdots, x_n) \quad \text{such that} \quad x_i \in \mathbb{R}\}$ endowed with the operations definded by:
For $x = (x_1, \cdots, x_n), \quad y = (y_1, \cdots, y_n) \in \mathbb{R}^n$ and $\lambda \in \mathbb{F}$

$$x + y = (x_1 + y_1, \cdots, x_n + y_n), \quad \lambda x = (\lambda x_1, \cdots, \lambda x_n).$$

One can show that these two operations satisfy the eight properties above so that $(\mathbb{R}^n, +, \cdot)$ is a vector space.

Example 1.3. Let $X = M_{n\times m}$ be the set of all $n \times m$-matrices with n rows and m columns, with the following operations:

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots a_{1m} \\ a_{21} & a_{22} & \ldots a_{2m} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots a_{nm} \end{pmatrix} B = \begin{pmatrix} b_{11} & b_{12} & \ldots b_{1m} \\ b_{21} & b_{22} & \ldots b_{2m} \\ \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \ldots b_{nm} \end{pmatrix}$$

$$A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{21} & \ldots a_{1m} + b_{1m} \\ a_{21} + b_{21} & a_{22} + b_{22} & \ldots a_{2m} + b_{2m} \\ \vdots & \vdots & \vdots \\ a_{n1} b_{n1} & a_{n2} + b_{21} & \ldots a_{nm} + b_{nm} \end{pmatrix} \lambda A = \begin{pmatrix} \lambda a_{11} & \lambda a_{12} & \ldots \lambda a_{1m} \\ \lambda a_{21} & \lambda a_{22} & \ldots \lambda a_{2m} \\ \vdots & \vdots & \vdots \\ \lambda a_{n1} & \lambda a_{n2} & \ldots \lambda a_{nm} \end{pmatrix}$$

where $A, B \in M_{n\times m}$. We can show that $(M_{n\times m}, +, \cdot)$ is a vctor space, usually called the **vector space of the $n \times m$-matrices.**

Example 1.4. Let $X = C[a, b]$ be the set of all continuous functions defined on the interval $[a, b]$. We endow $C[a, b]$ with the following operations:
$\forall f, g \in C[a, b]$ and $\forall \lambda \in \mathbb{R}$ or $\mathbb{C}$

1. $(f + g)(t) = f(t) + g(t)$

2. $(\lambda f)(t) = \lambda f(t)$.

One can show that $(C[a,b], +, \cdot)$ is vector space usually called the **space of the continuous functions on** $[a,b]$**.**

Chapter 2

Metric and Normed Spaces

Definition 2.1. *Let X be an arbitrary non-empty set. A metric on X is a mapping $d : X \times X \longrightarrow \mathbb{R}$ satisfying*

1. $d(x,y) \geq 0$ *and* $d(x,y) = 0$ *if and only if* $x = y$.
2. $d(x,y) = d(y,x) \quad \forall x,y \in X$
3. $d(x,z) \leq d(x,y) + d(y,z) \quad \forall x,y,z \in X$

The set X endowed with the mapping d and sastisfying these three properties and denoted by (X,d) is called a **metric space.**

Remark 2.1. *Two different metrics defined on the same set X in general define two different metric spaces.*

Example 2.1. Let X be an arbitrary non-empty set X. Define on X the following metric:

$$d : X \times X \longrightarrow \mathbb{R} \quad (x,y) \longmapsto d(x,y) = \begin{cases} 1 \text{ if } x \neq y \\ 0 \text{ if } x=y \end{cases}$$

(X,d) is a metric space, usually called the **trivial metric space**.

Remark 2.2. *This example tells us that any non-empty set can be made into a metric space.*

Example 2.2. Let $X = \mathbb{R}^n, x = (x_1, \cdots, x_n), \quad y = (y_1, \cdots, y_n)$. Define the following metrics on $\mathbb{R}^n$.

$$\begin{aligned} d_1(x,y) &= |x_1 - y_1| + |x_2 - y_2| + \cdots + |x_n - y_n| \\ d_2(x,y) &= \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2 + \cdots + |x_n - y_n|^2} \\ d_\infty(x,y) &= \max_{1 \leq i \leq n} |x_i - y_i| \end{aligned}$$

One can show that d_1, d_2, d_∞ are metrics and $(\mathbb{R}^n, d_1), (\mathbb{R}^n, d_2), (\mathbb{R}^n, d_\infty)$ are metric spaces.

Example 2.3. Let $X = C[a,b]$ be the set of all continuous functions defined in the interval $[a,b]$; define the following metric on X:

$$d_\infty(f,g) = \max_{a \leq t \leq b} |f(x) - g(x)|$$

$(C[a,b], d_\infty)$ is a metric space and known as the metric space of the continous functions with the supremum metric.

Example 2.4. Let $X = C[a,b]$, endowed with the folllowing metric.

$$d_1(f,g) = \int_a^b |f(t) - g(t)| dt$$

where the integral is in the sense of Riemann. $(C[a,b], d_1)$ is a metric space.

Note that $(C[a,b], d_1)$ and $(C[a,b], d_\infty)$ are two distinct metric spaces.

Definition 2.2. *A sequence in a metric space (X,d) is a mapping from the natural numbers $\mathbb{N}$ into X, usually denoted by (x_n).*
(x_n) is said to be a convergent sequence to $x \in X$ if

$$\forall \epsilon > 0, \quad \exists N = N(\epsilon) \in \mathbb{N} \quad | \quad n \geq N \Longrightarrow d(x_n, x) \leq \epsilon$$

.

Definition 2.3. *A sequence (x_n) in a metric space (X,d) is said to be a Cauchy sequence if*

$$\forall \epsilon > 0 \quad \exists N = N(\epsilon) \in \mathbb{N} \quad | \quad n, m \geq N \Longrightarrow d(x_n, x_m) \leq \epsilon$$

Remark 2.3. *It is a simple exercise to show that any convergent sequence in (X,d) is a cauchy sequence.*

Remark 2.4. *A Cauchy sequence (x_n) in (X,d) is not necessarily a convergent sequence in (X,d)*

Example 2.5. Let $X = (0,1)$ and define a metric on X by $d(x,y) = |x-y|$ where the bars represent the absolute value Let $x_n = \frac{1}{n}$. It is obvious that (x_n) is Cauchy sequence in (X,d), that is $d(x_n, x_m) = |\frac{1}{n} - \frac{1}{m}| \longrightarrow 0$ as $n, m \longrightarrow \infty$ but $x_n \longrightarrow 0$ where $0 \notin (0,1)$

Definition 2.4. *A metris space (X,d) is said to be a complete metric space if every Cauchy sequence in (X,d) is convergent.*

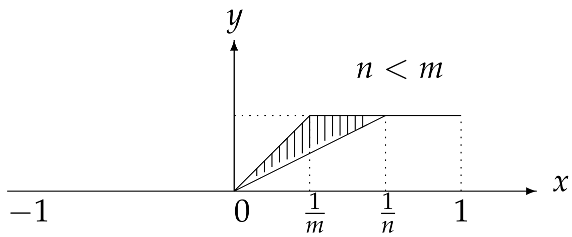
Example 2.6. The space in the Example 2.5 above is not complete.

Example 2.7. Let $C[-1,1]$ an define the metric

$$d(f,g) = \int_{-1}^{1} |f(t) - g(t)| dt$$

$C[-1,1]$ is not a complete metric space. To prove that, let's define the following sequence.

$$f_n(x) = \begin{cases} 0 & \text{if } -1 \leq x \leq 0 \\ nx & \text{if } 0 \leq x \leq \frac{1}{n} \\ 1 & \text{if } \frac{1}{n} \leq x \leq 1 \end{cases}$$

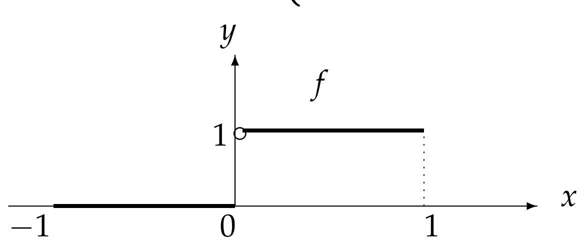


$$
\begin{aligned}
d(f_n, f_m) &= \int_{-1}^{1} |f_n(x) - f_m(x)| dx \\
&= \int_{0}^{\frac{1}{m}} (mx - nx) dx + \int_{\frac{1}{m}}^{\frac{1}{n}} (1 - nx) dx \\
&= \frac{m-n}{2} x^2 \Big|_0^{\frac{1}{m}} + \left(\frac{1}{n} - \frac{1}{m}\right) - \frac{n}{2} x \Big|_{\frac{1}{m}}^{\frac{1}{n}} \\
&= \frac{m-n}{2n^2} + \frac{1}{n} - \frac{1}{m}\left(\frac{1}{n^2} - \frac{1}{m^2}\right) \longrightarrow 0 \quad \text{as} \quad n, m \longrightarrow \infty.
\end{aligned}
$$

Note that $d(f_m, f_n)$ is the shaded area, so geometrically we can see that this area tends to 0 as $n, m \longrightarrow \infty$.

As $d(f_n, f_m) \longrightarrow 0$ as $n, m \longrightarrow \infty$, this implies that $f_n(x)$ is a Cauchy sequence in $(C[-1, 1], d)$, but we can see that $f_m(x)$ converges to the function

$$
f(t) = \begin{cases} 0 & \text{if } -1 \leq x \leq 0 \\ 1 & \text{if } 0 < x < 1 \end{cases}
$$



$f \notin C[-1,1]$ since f is not continuous at $x = 0$, even though

$$d(f_m, f) = \int_{-1}^{1} |f_m(x) - f(x)| dx = \frac{1}{2} \cdot \frac{1}{m} \longrightarrow 0$$

Example 2.8. In $C[-1,1]$ define the metric given by

$$d(f,g) = \max_{-1 \leq x \leq 1} \{|f(x) - g(x)|\}$$

It' not hard to show that $C[-1,1]$ endowed with this metric d is a complete metric space.

Example 2.9. Let $\mathbb{R}^n$ endowed with any of the metrics d_1, d_2 and d_∞ in the example 2.2, then $(\mathbb{R}^n, d_1), (\mathbb{R}^n, d_2)$ and $(\mathbb{R}^n, d_\infty)$ are examples of complete metric spaces. Note that later one we will show that if we endow $\mathbb{R}^n$ with any metric ρ, then $(\mathbb{R}^n, \rho)$ is complete metric space.

Definition 2.5. *Let $(X, +, \cdot)$ be a vector space over the scalar field $F(\mathbb{R}$ or $\mathbb{C})$. A **norm** in X is a mapping denoted by $|| \cdot ||_X$ from X into $[0, \infty)$, that is*

$$|| \cdot ||_X : X \longrightarrow [0, \infty)$$

satisfying the conditions

1. $||x||_X = 0$ *if and only if* $x = 0$
2. $||\lambda x||_X = |\lambda| ||x||_X, \forall x \in X, \quad \lambda \in F$
3. $||x + y||_X \leq ||x||_X + ||y||_X, \forall x, y \in X$

A vector space $(X, +, \cdot)$ endowed with a norm $|| \cdot ||_X$ is called normed vector space, usually denoted by $(X, || \cdot ||_X)$ or $(X, +, \cdot, || \cdot ||_X)$

Remark 2.5. *The condition 3 is usually called the **triangle inequality**.*

Remark 2.6. *Any normed vector space $(X, || \cdot ||_X)$ can be made into a metric space, by defining the following metric $d(x, y) = ||x - y||_X$*

Remark 2.7. *Under certain conditions, a metric space (X, d) ca be made into normed space.*

Example 2.10. Let $X = \mathbb{R}^n$. Define in X the following mappings. For $x = (x_1, \cdots, x_n)$

$$||x||_1 = |x_1| + \cdots |x_n|$$

$$||x||_2 = \sqrt{x_1^2 + \cdots + x_n^2}$$

$$||x||_\infty = \max\{|x_1|, \cdots, |x_n|\}$$

With some degree of difficulty, one can show that $||\cdot||_1, ||\cdot||_2$ and $||||_\infty$ satisfy all three conditions of norms, so that $(\mathbb{R}^n, ||\cdot||_1)$, $(\mathbb{R}^n, ||\cdot||_2)$ and $(\mathbb{R}^n, ||\cdot||_\infty)$ are normed spaces.

Example 2.11. Let $X = C[a, b]$, the continuous functions defined in the interval $[a, b]$. Define the mapping

$$||f||_\infty = \max_{a \leq x \leq b} \{|f(x)|\}$$

One can show that $||\cdot||_\infty$ is a norm and $(C[a, b], ||\cdot||_\infty)$ is a normed space.

Example 2.12. Let $C[a, b]$. Define the following mappings

$$||f||_1 = \int_a^b |f(t)| dt$$

$$||f||_2 = \left(\int_a^b |f(t)|^2 dt \right)^{\frac{1}{2}}$$

$$||f||_p = \left(\int_a^b |f(t)|^p dt \right)^{\frac{1}{p}}$$

We can show that $||\cdot||_1, ||\cdot||_2$ and $||\cdot||_p$, $(1 < p < \infty)$ are norms in $C[a, b]$ and the integrals are taken in the Riemann sense.

Note that except for $||\cdot||_1$, the proofs of the properties of norms for $||\cdot||_p \quad (1 < p < \infty)$ is not very simple.

Example 2.13. Let $X = \mathbb{R}^n, x = (x_1, \cdots, x_n)$ and for $1 < p < \infty$, define the mappings $||\cdot||_p$ by

$$||x||_p = \left(|x_1| + \cdots + |x_n|^p\right)^{\frac{1}{p}} = \Big(\sum_{i=1}^{n}|x_i|^p\Big)^{\frac{1}{p}}$$

Again one can prove that $||\cdot||_p$ is a norm and so $(\mathbb{R}^n, ||\cdot||_p)$ is a normed space usually denoted by l_p^n.

Example 2.14. The spaces l_p for $(1 \leq< p \leq \infty)$. A sequence $x = (x_n)$ belongs to the space l_p for $(1 \leq p \leq \infty)$ and $x_i's$ in $\mathbb{R}$ if

$$||x||_p = \Big(\sum_{i=1}^{\infty}|x_i|^p\Big)^{1/p} < \infty \quad \text{for} \quad (1 \leq< p < \infty)$$

$$||x||_\infty = \operatorname*{Sup}_{n\geq 1}|x_n| < \infty$$

.

Define the following operations in l_p for $x = (x_1, \cdots, x_n, \cdots)$ and $y = (y_1, \cdots, y_n, \cdots)$

$$x + y = (x_1 + y_1, \cdots, x_n + y_n, \cdots)$$

$$\lambda x = (\lambda x_1, \cdots, \lambda x_n, \cdots)$$

The mappings $||x||_p$ and $||x||_\infty$ satisfy the propoerties of norms. Therefore we can show that $(l_p, +, \cdot, ||\cdot||_p)$ is a normed vector space. Note that these spaces l_p for $1 \leq p \leq \infty$ are usually known as **little l_p spaces**.

Remark 2.8. *To show that $||\cdot||_p$ is a norm for $1 < p < \infty$, we have to use the well-known "Holder's Inequality" that says:*
If $x = (x_1, \cdots, x_n, \cdots) \in l_p$ and $y = (y_1, \cdots, y_n, \cdots) \in l_q$, $1 < p, q < \infty, \dfrac{1}{p} + \dfrac{1}{q} = 1$, then

$$\sum_{n=1}^{\infty}|x_n y_n| \leq \left(\sum_{n=1}^{\infty}|x_n|^p\right)^{1/p}\left(\sum_{n=1}^{\infty}|y_n|^q\right)^{1/q}$$

Example 2.15. Let $X = C^1[a, b]$ the set of real-valued functions that is continuous differentialble functions in the interval $[a,b]$. Define the mapping $||\cdot||$ on $C^1[a,b]$ by

$$||f|| = \max_{a\leq x\leq b}\{|f(t)|\} + \max_{a\leq t\leq b}\{|f'(t)|\}$$

One can show that with the usual operations of additions and scalar multiplication of functions, $(C^1[a,b],+,\cdot)$ is a vector space and that $||\cdot||$ is a norm in $C^1[a,b]$ and so $(C^1[a,b],+,\cdot,||\cdot||)$ is a normed space.

Example 2.16. Define the space $C = \{x = (x_n);\ \lim_{n\longrightarrow\infty} x_n \quad \text{exists}\}$ and a function $||\cdot||_C$ by

$$||x||_C = \operatorname*{Sup}_{n\geq 1}|x_n|$$

Define the operations by for $x = (x_n), y = (y_n)$

$$x + y = (x_n + y_n)$$

$$\lambda x = (\lambda x_n)$$

one can show that $||\cdot||_C$ is a norm and $(C,+,\cdot)$ is a vector space and so $(C,+,\cdot,||\cdot||_C)$ is a normed space.

Remark 2.9. *The space C is well-known as the* ***space of convergent sequences****.*

Example 2.17. Define the space $C_0 = \{x = (x_n);\ \lim_{n\longrightarrow\infty} x_n = 0\}$ and a function $||\cdot||_{C_0}$ by

$$||x||_{C_0} = \operatorname*{Sup}_{n\geq 1}|x_n|$$

$(C_0,+,\cdot,||\cdot||_{C_0})$ is normed vector space and is well known as the **space of convergent sequences to zero**.

Remark 2.10. *Note that we have the following inclusions of the spaces l_∞, C and C_0: $C_0 \subsetneq C \subsetneq l_\infty$.*

Definition 2.6. *Define the space $BV[a,b]$ in the following way.*
Let $a = x_0 \leq x_1 \leq \cdots \leq x_n = b$ be a partition of the interval $[a,b]$ and

$$V_{[a,b]}(f) = Sup\sum_{i=1}^{n}|f(x_i) - f(x_{i-1})|$$

for all partitions and f a real-valued function defined in $[a,b]$.
Then

$$f \in BV[a,b] \quad \textit{if and only if} \quad V_{[a,b]}(f) < \infty.$$

Define

$$||f||_{BV} = |f(a)| + V_{[a,b]}(f)$$

One can show that with the usual operations of additions and scalar multiplication of functions that $(BV[a,b],+,\cdot)$ is a vector space and $||\cdot||_{BV}$ is a norm on it so that $(BV[a,b],+,\cdot,||\cdot||_{BV})$ is normed space.

Remark 2.11. *The space $BV[a,b]$ is well-known as the space on the **Bounded variations functions**.*

Definition 2.7. *Define the space $AC[a,b]$ as the following:*
A function $f \in BV[a,b]$ is said to be absolutely continuous if for each $\epsilon > 0$, there is $\delta > 0$ such that $\sum_{i=1}^{n} |f(b_i) - f(a_i)| < \epsilon$ whenever $(a_i,b_i), i = 1,\cdots,n$ are non-overlaapping subintervals of $[a,b]$ and $\sum_{n=1}^{n} |b_i - a_i| < \delta$.

The space $AC[a,b]$ is defined for intervals $[a,b]$ and consists of all absolutely continuous functions on $[a,b]$. We endow $AC[a,b]$ with

$$||f||_{AC} = |f(a)| + V_{[a,b]}(f)$$

One can show that with the usual operations, $(AC[a,b],+,\cdot)$ is a vector space and $||\cdot||_{AC}$ is norm. $AC[a,b]$ is called the **space of absolutely continous functions**.

Example 2.18. Let $H^p(\mathbb{D})$ be the set of analytic functions F defined in the unit disc $\mathbb{D}$ such that

$$||F||_{H^p} = \operatorname*{Sup}_{0<r\leq 1}\left(\int_{-\pi}^{\pi} |F(re^{i\theta})|^p d\theta\right)^{\frac{1}{p}} < \infty \quad \text{for} \quad 1 \leq p < \infty$$

and

$$||F||_{H^\infty} = \operatorname*{Sup}_{|z|<1}|F(z)| < \infty, \quad \text{for} \quad p = \infty$$

One can show that $(H^p(\mathbb{D}), +, \cdot, ||\cdot||_{H^p})$ is a normed vector space. This space is well-known as the **Hardy space**.

Example 2.19. Let $X = \text{Lip}_\alpha, 0 < \alpha \leq 1$, be the space of all continous functions f defined in the interval $[a, b]$ such that

$$||f||_{\text{Lip}_\alpha} = ||f||_\infty + \underset{\substack{x \\ h>0}}{\text{Sup}} \left| \frac{f(x+h) - f(x)}{h^\alpha} \right| < \infty$$

$(\text{Lip}_\alpha, +, \cdot, ||\cdot||_{\text{Lip}_\alpha})$ is normed vector space and is well-known as the **Lipschitz space**.

Example 2.20. Let $X = \Lambda_\alpha, 0 < \alpha < 2$ be the space of the continuous functions in the interval $[a, b]$ such that

$$||f||_{\Lambda_\alpha} = ||f||_\infty + \underset{\substack{x \\ h>0}}{\text{Sup}} \left| \frac{f(x+h) + f(x-h) - 2f(x)}{h^\alpha} \right| < \infty$$

$(\Lambda_\alpha, +, \cdot, ||\cdot||_{\Lambda_\alpha})$ is normed vector space and is well-known as the **Generalized Lipschitz space**.

Remark 2.12. *Some properties of the spaces Lip_α and Λ_α for $0 < \alpha < 2$*

1. *For $\alpha = 1, \Lambda_1$ is well-known to be the **Zygmund class** in the theory of approximations and usually denoted by $\Lambda_\star$.*

2. *for $\alpha > 1, Lip_\alpha$ only contains the constant functions.*

3. $\Lambda_1 \subsetneq Lip_\beta \subsetneq Lip_\alpha, \quad \alpha < \beta.$

4. $Lip_\alpha \subsetneq \Lambda_\alpha, \quad 0 < \alpha < 1.$

5. $Lip_\alpha = \Lambda_\alpha, \quad 0 < \alpha < 1$

Definition 2.8. *Define the space S_α of all analytic functions F defined in the unit disc such that*

$$||F||_{S_\alpha} = \underset{0<|z|<1}{Sup} (1-|z|)^{1-\alpha}|F'(z)| < \infty, \quad 0 < \alpha < 1$$

The space $(S_\alpha, +, \cdot, ||\cdot||_{S_\alpha})$ *is a normed vector space and is well-known as the analytic characterization of the space* Lip_α *for* $0 < \alpha < 1$ *on* $[0, 2\pi]$.

Example 2.21. Define the space BMO as the space of all periodic functions f of period 2π and defined in $[0, 2\pi]$ such that

$$\begin{aligned} ||f||_{BMO} &= ||f||_1 + \operatorname*{Sup}_I \frac{1}{|I|}\int_{|I|} |f(t) - f_I| dt < \infty \\ ||f||_1 &= \int_0^{2\pi} |f(t)| dt \end{aligned}$$

I is an interval in $[0, 2\pi]$ and $f_I = \frac{1}{|I|}\int_I f(t)dt$
$(BMO, +, \cdot, ||\cdot||_{BMO})$ is a normed vector space and is well-known in the area of Harmonic Analysis as the space of **Bounded Means Oscillations**.

Example 2.22. De Souza's Spaces B^1.
Let f be a function defined in the interval $[a, b]$ that can be represented in the form

$$f(t) = \sum_{n=0}^{\infty} c_n b_n(t)$$

where c_n's are numbers such that $\sum_{n=0}^{\infty} |c_n| < \infty$ and the b_n's are functions of the form

$$b_n(t) = \frac{1}{|I_n|}\Big[\chi_{Ln}(t) - \chi_{R_n}(t)\Big], \quad b_0(t) = 1$$

where the I_n's are interval in $[a, b]$.
χ_{R_n}, χ_{L_n} are the characteristic functions of the interval R_n and L_n, $I_n = R_n \bigcup L_n$ with R_n and L_n the halves of the interval I_n.

$||f||_{B^1} = \operatorname{Inf} \sum_{n=0}^{\infty} |c_n|$, where th infimum is taken over all possible representations of f. The space $(B^1, +, \cdot, ||\cdot||_{B^1})$ is a normed vector space, usually mentioned in the area of harmonic analysis as the **De Souza Space**.

Remark 2.13. *The space* B^1 *now well-known as the De Souza's space was introduced by Geraldo De Souza in his PhD dissertation in 1980 in the department of mathematics at the University of Albany, the capital city of the New-York state, under the supervision of professor*

Richard O'Neil.

Example 2.23. The complex form of the De Souza Space.
Let A be the space of analytic functions defined in the unit disc $\mathbb{D}$ such that

$$||F||_A = \int_0^1 \int_0^{2\pi} |F'(re^{i\theta})| d\theta dr < \infty$$

where the dash means the derivative of F. The space $(A, +, \cdot, || \cdot ||_A)$ is normed vector space.

Remark 2.14. *The space A is known as the analytic characterization of the De Souza's space B^1 and it was published in the journal of London Mathematicaal Society in 1983. It was a very old unsolved problem to characterize the boundary values of a function in the space A, in other words, characterize the space of functions defined on the inteval $[0, 2\pi]$ that are limit of the real part of these analytic functions in A.*

Chapter 3

Banach and Hilbert Spaces

Let $(X, +, \cdot, \|\cdot\|_X)$ be a normed vector space. We always look $\|x\|_X$ as the length of the vector x. As we saw earlier that $d(x, y) = \|x - y\|_X$, defines a metric on X and we usually assume that the normed space X is endowed with this metric and the topology associated with this is called the **metric topology**. Therefore, a sequence (x_n) in X which converges to $x \in X$ in the norm topology if and only if $\|x_n - x\|_X \longrightarrow 0$ as $n \longrightarrow \infty$ that is,

$$x_n \longrightarrow x \quad \text{in the norm topology} \quad \Longleftrightarrow \|x_n - x\|_X \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty$$

Similarly, we say that a sequence (x_n) in X is a Cauchy sequence in the norm topology if $\|x_n - x_m\|_X \longrightarrow 0$ as $n, m \longrightarrow \infty$. More precisely,

$$\forall \epsilon > 0, \exists N = N(\epsilon) \in \mathbb{N} \quad \text{such that if} \quad n, m \geq N \quad \text{then} \quad \|x_n - x_m\|_X \leq \epsilon$$

Definition 3.1. *A normed vector space $(X, +, \cdot, \|\cdot\|_X)$ is said to be a Banach space if every Cauchy sequence in X is convergent in X.*

Remark 3.1. *An equivalent way to define a Banach space is that the associated metric space to the normed vector space is complete, that is (X, d) where $d(x, y) = \|x - y\|_X$ is a complete metric space.*

Remark 3.2. *The concept of completeness in a normed space is very important in functional analysis and the varieties of normed space which are complete in analysis is enormous. It does not mean that one is more important that the others, all that depends on the area we are working on and what our goals are.*

Remark 3.3. *The completeness of a normed vector space X depends on the norm we define on it. the same space can be complete with one norm but not complete with another.*

Example 3.1. Let's get back to the Example 2.7 and 2.8.

In $C[-1,1]$ we define the norm $\|f\|_\infty = \max_{a\le x\le b}\{|f(x)|\}$.
One can show that $(C[-1,1],+,\cdot,\|\cdot\|_\infty)$ is a complete normed space,that is a Banach space. On the other hand, if we endow $C[-1,1]$ with the norm $\|f\|_1 = \int_{-1}^{1}|f(t)|dt$, then we can show that the sequence

$$f_n(x) = \begin{cases} 0 & \text{if } -1\le x\le 0 \\ nx & \text{if } 0\le x\le \frac{1}{n} \\ 1 & \text{if } \frac{1}{n}\le x\le 1 \end{cases}$$

is a Cauchy sequence in $(C[-1,1],+,\cdot,\|\cdot\|_1)$, that is $\|f_n - f_m\|_1 \longrightarrow 0$ as $n,m\longrightarrow\infty$. Moreover, we can show that $f_n(x)$ converges to $f(x)$ in the norm topology where $f(x)=0$ if $-1\le x<0$, $f(x)=1$ if $0<x\le 1$. Indeed we showed that

$$\|f_n - f\|_1 = \int_{-1}^{1}|f_n(t)-f(t)|dt = \frac{1}{2}\cdot\frac{1}{n}\longrightarrow 0 \quad \text{as} \quad n\longrightarrow\infty$$

The conclusion is that the sequence (f_n) is a Cauchy sequence in $(C[-1,1],+,\cdot,\|\cdot\|_1)$ which converges to f but f is not a function in $C[-1,1]$ since it's not continuous at $x=0$. Therefore $(C[-1,1],+,\cdot,\|\cdot\|_1)$ is not a Banach space.

Remark 3.4. *The lack of completeness in a normd vectors space is not all bad. In reality, each metric space(complete or not) can be viewed as a "dense" subspace in some complete metric space. This complete metric space is usually called the* ***complettion of a metric space****.*

Example 3.2. $(\mathbb{R}^n,\|\cdot\|_1),(\mathbb{R}^n,\|\cdot\|_2)$ and $(\mathbb{R}^n,\|\cdot\|_\infty)$ in example 2.1 are Banach spaces.

Remark 3.5. *Examples 2.11-2.25 are examples of Banach spaces.*

Remark 3.6. *Example 3.2 and the examples in the remark 3.5 are Banach spaces quite often seen in various areas of analysis.*

Remark 3.7. *Note that in a Banach space, the concept of convergent sequence and Cauchy sequence are equivalent, therefore it gives us the luxury to know if a given sequence is convergent just comparing the terms of the sequence for n,m very large, that is for $n,m\longrightarrow\infty$.*

Example 3.3. An important example of a normed space not complete is the rational numbers $\mathbb{Q}$ endowed with the usual operations of addition and multiplication ($F =$

$\mathbb{Q}$) and a norm defined as the absolute value, that is for $q \in \mathbb{Q}$, $\|q\|_{\mathbb{Q}} = |q|$.
$(\mathbb{Q}, +, \cdot, \|\cdot\|_{\mathbb{Q}})$ is a normed space over the field $F = \mathbb{Q}$. But it's not complete.
For example, take $q_{n+1} = \dfrac{1}{2}(q_n + \dfrac{2}{q_n}), \quad q_n \in \mathbb{Q}$. Note that (q_n) is a Cauchy sequence in $\mathbb{Q}$; moreover $q_n \longrightarrow \sqrt{2} \notin \mathbb{Q}$
the completion of $\mathbb{Q}$ is the well-known set of real numbers.

Remark 3.8. *A technique to how a normed space is complete or incomplete is the use of the concept of absolutely convergent series in a normed space which generalize the concept of absolutely convergent series of real numbers. For example, if (x_n) are real numbers, then every absolutely convergent series is convergent.*

Definition 3.2. *Let X be a normed vector space and $\sum_{n=1}^{\infty} x_n, \quad x_n \in X$ a series in X. We say that the series $\sum_{n=1}^{\infty} x_n, \quad x_n \in X$ converges absolutely if the numerical series $\sum_{n=1}^{\infty} \|x_n\|_X$ is convergent.*

Remark 3.9. *The concept of series in a normed vector space X can be defined as: Let $x_1, \cdots, x_n, \cdots \in X$ and $S_n = x_1 + \cdots + x_n$ the partial sum. We say that*

$$S_n \longrightarrow x \quad \text{as} \quad n \longrightarrow \infty \quad \text{in the norm topology if} \quad \left\| \sum_{i=1}^{n} x_i - x \right\|_X \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty$$

, and so we say that the series $\sum_{i=1}^{\infty} x_i$ converges to x

The next result is very important and useful to show that a given normed space is complete or incomplete.

Theorem 3.1. *Let X be a normed vector space and $\sum_{n=1}^{\infty} x_n$ a series in X.
X is complete if and only if every absolutely convergent series in X is convergent, that is*

$$X \quad \text{is complete} \quad \Longleftrightarrow \quad \text{every absolutely convergent series in } X \text{ is convergent}$$

Remark 3.10. *This theorem appears in any calculus book in section of numerical series in the following form:*

"Any numerical series that is absolutely convergent is convergent"

This is true since the real numbers $\mathbb{R}$ as a normed vector space is complete but the difficult part is to show that if any numerical series that is absolutely convergent is convergent, then the real numbers as a normed space is complete.

Remark 3.11. *To show Theorem 3.1, we need the following Theorem.*

Theorem 3.2. *Let X be a normed vector space. If (x_n) is a Cauchy sequence in X which contains a convergent subsequence (x_{n_k}), then (x_n) is a convergent sequance in X.*

Proof. Let x_n be a Cauchy sequence in X and (x_{n_k}) a convergent subsequence, i.e $x_{n_k} \longrightarrow x$ for $k \longrightarrow \infty$ or $\|x_{n_k} - x\|_X \longrightarrow 0$ as $k \longrightarrow \infty$.
Writing $x_n - x = x_n - x_{n_k} + x_{n_k} - x$ and using the triangle inequality, we get:

$$\|x_n - x\|_X \leq \|x_n - x_{n_k}\|_X + \|x_{n_k} - x\|_X$$

Now since (x_{n_k}) is convergent, we can find an N_1 so that for $n_k \geq N_1$, $\|x_{n_k} - x\|_X \longrightarrow 0$
On the other hand (x_n) is a Cauchy sequence, so we can find an N_2 so that for $n, n_k \geq N_2$, $\|x_n - x_{n_k}\|_X \longrightarrow 0$. Consequently, for $n > \text{Max}\{N_1, N_2\}$, we get $\|x_n - x\|_X \longrightarrow 0$ □

Proof. Proof of Theorem 3.1
$\Longrightarrow$) Let (x_n) be a sequence so that $\sum_{n=1}^{\infty} \|x_n\|_X < \infty$. Define a sequence $S_n = \sum_{k=1}^{n} x_k$, then $S_n - S_m = \sum_{k=n}^{m} x_k$. Assume, without loss of generality, that $n < m$ and so

$$\|s_n - s_m\|_X \leq \sum_{k=n}^{m} \|x_k\|_X$$

and as $\sum_{k=1}^{\infty} \|x_k\|_X < \infty$, we have

$$\|S_n - S_m\|_X \longrightarrow 0 \quad \text{as} \quad n, m \longrightarrow \infty$$

That is $\sum_{n=1}^{\infty} x_n$ is convergent.
$\Longleftarrow$) The idea of the proof is to take a Cauchy sequence in X and show using the hypothesis that we can construct a convergent subsequence and so the Theorem 3.2

concludes that this Cauchy sequence is convergent.
Let (x_n) be a Cauchy sequence in X, that is given ϵ, $\exists N = N(\epsilon)$ so that $\|x_n - x_m\|_X \leq \epsilon$ for all $n > N$
Let's take $\epsilon = \frac{1}{2}$ and since (x_n) is a Cauchy sequence. There is an N_1 and so that $\|x_n - x_m\|_X < \frac{1}{2}, n, m \geq N_1$.
For $\epsilon = \frac{1}{2^2}$ and since (x_n) is a Cauchy sequence so there is an N_2 so that $N_1 \leq N_2$ and $\|x_n - x_m\| \leq \frac{1}{2^2}, n, m \geq N_2$.
For $\epsilon = \frac{1}{2^3}$ and since (x_n) is a Cauchy sequence, there is an N_3 so that $N_1 \leq N_2 \leq N_3$ and $\|x_n - x_m\|_X \leq \frac{1}{2^3}, n, m \geq N_3$.
If we continue this procedure, we get for $\epsilon = \frac{1}{2^k}$, we will find an N_k, so that $N_1 \leq N_2 \leq \cdots \leq N_k$ and $\|x_n - x_m\| \leq \frac{1}{2^k}$, for $n, m \geq N_k$.
We now consider the subsequence $x_{N_1}, x_{N_2}, \cdots, x_{N_k}, \cdots$. We will show that this subsequence (x_{N_k}) is convergent. In fact, definde $y_k = x_{N_k} - x_{N_{k-1}}$ if $k > 1$ and $y_1 = x_{N_1}$. We will show that $\sum\limits_{k=2}^{\infty} y_k$ is absolutely convergent i.e.

$$\sum_{k=2}^{\infty} \|y_k\|_X = \sum_{k=1}^{\infty} \|x_{N_k} - x_{N_{k-1}}\|_X \leq \sum_{k=2}^{\infty} \frac{1}{2^k} < \infty$$

Therefore using the hypothesis the series $\sum\limits_{k=1}^{\infty} y_k$ is convergent i.e. there is an $y \in X$ so that $\left|\left| \sum\limits_{k=2}^{n} y_k - y \right|\right|_X \longrightarrow 0$ as $n \longrightarrow \infty$. Note that

$$\sum_{k=2}^{n} y_k = \sum_{k=2}^{n} (x_{N_k} - x_{N_{k-1}}) = x_{N_2} - x_{N_1} + x_{N_3} - x_{N_2} + \cdots + x_{N_n} - x_{N_{n-1}} = x_{N_n}$$

so

$$\sum_{k=2}^{n} y_k - y = x_{N_n} - y \quad \text{i.e.} \quad \|x_{N_n} - y\|_X \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty$$

This tells us that the Cauchy sequence (x_n) in X has a convergent subsequence (x_{N_k}). Therefore by Theorem 3.2, it imply that (x_n) is convergent so that X is complete.

□

Remark 3.12. *Theorem 3.1 is possibly the easiest way to show that the De Souza's space in* *the example 2.25 is a complete and so a Banach.*

Example 3.4. The Lebesgue spaces $L_p, 0 < p \leq \infty$. The L_p is a space defined as the space of all measurable functions f on a measure space $(X, \mathcal{F}, \mu)$ for which $|f(x)|^p$ is integrable in other words

$$L_p = \{f : X \longrightarrow \mathbb{R} \quad \text{or} \quad \mathbb{C} \quad \text{so that} \quad \int_X |f(x)|^p d\mu(x) < \infty\}$$

We define a "norm" in L_p as

$$\|f\|_p = \left(\int_X |f(x)|^p d\mu(x) \right)^{\frac{1}{p}} \quad \text{for} \quad 0 < p < \infty$$

$$\|f\|_\infty = \text{EssSup} f = \text{Inf}\{M; |f(x) < M| \quad \text{a.e.}\}$$

The L_p spaces is not a space of functions since functions which agree almost everywhere have the same integral. Indeed L_p spaces is a space formed by equivalence classes. In fact the class $[f]$ is defined as $[f] = \{g : g = f \text{a.e.}\}$ so if $h, g \in [f]$, we have $\|f\|_p = \|g\|_p = \|h\|_p$

Remark 3.13. *L_p is endowed with the usual operations of adition and scalar multiplication and L_p is closed under these operations To see the closedness of addition, we need the following fact:*

$$|f+g|^p \leq (|f| + |g|)^p \leq (2Max\{|f|, |g|\})^p = 2^p Max\{|f|^p, |g|^p\} \leq 2^p(|f|^p + |g|^p)$$

Therefore if $f, g \in L_p$, then $f + g \in L_p$
For the scalar multiplication, it follows easily from the fact that $|\lambda f|^p = |\lambda|^p |f|^p$

Remark 3.14. *For $1 \leq p \leq \infty$, we can show that $\|f\|_p$ is a semi-norm, that is, it satisfies all the properties of norm except $\|f\|_p = 0$ is not necessarily implies $f = 0$. Indeed, we have that if $\|f\|_p = 0$ then $f = 0$ almost everywhere which is, the set for which $f \neq 0$ has measure zero. Therefore, to resolve this issue, we identify all the functions whose difference is a constant i.e. $f, f + c$ are considered the same. This identification can be formally done using the quotient space defined with the equivalence relations $f \sim 0$ if $\mu\{x \in X : f(x) \neq 0\} = 0$ that is the measure of the set $\{x \in X : f(x) \neq 0\}$ is zero. This quotient space which is formed by* *equivalent classes is called the **Lebesgue space** L_p.*

Remark 3.15. *For $0 < p < 1$, $\|f\|_p$ is not a norm since $\|f+g\|_p > \|f\|_p + \|g\|_p$ but if we define the function $d(f,g) = \|f-g\|_p^p, \quad 0 < p < 1$, we can show that d is a metric in L_p and that (L_p, d) is a complete metric space.*

Remark 3.16. *In the spaces L_p, we have two important inequalities that we state here:*

Theorem 3.3. ***1. Minkowski Inequality***

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p \quad \textit{for all} \quad f,g \in L_p, 1 \leq p \leq \infty$$

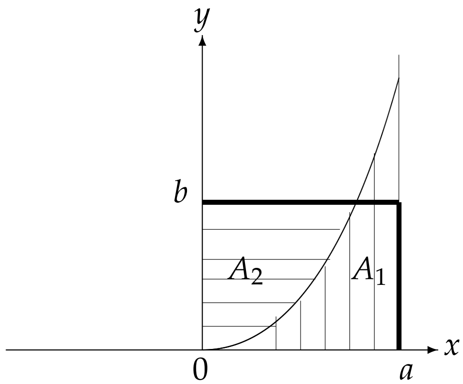
2. Holder's Inequality

$$\textit{If} \quad f \in L_p, g \in L_q, 1 \leq p,q \leq \infty, \frac{1}{P} + \frac{1}{q} = 1 \quad \textit{then} \quad f,g \in L_1.$$

$$\textit{Moreover,} \quad \|fg\|_1 \leq \|f\|_p \|g\|_q$$

Remark 3.17. *The proof of Theorem 3.3 follows from a simple lemma established for real numbers which says :*
If a,b are positives real numbers such that $1 < p,q < \infty$, then $ab \leq \dfrac{a^p}{p} + \dfrac{b^q}{q}$ and the equality holds if $a = b$
For $p = \infty$, the theorem is immediate.

The proof of this assertion can easily be done if we use geometric considerations. For example, define the functions $\varphi(t) = t^{p-1}, 0 \leq t \leq a$ whose inverse is $\varphi^{-1}(t) = t^{q-1}, 0 \leq t \leq b$ for $\dfrac{1}{p} + \dfrac{1}{q} = 1$



$A_1 = \int_0^a \varphi(t)dt = \int_0^a t^{p-1}dt = \dfrac{a^p}{p}$
$A_2 = \int_0^b \varphi^{-1}(t)dt = \int_0^b t^{q-1}dt = \dfrac{b^q}{q}$

Geometrically, we can easily se that the area of the rectangle of dimensions a and b is smaller than the area A_1 plus area A_2. That is $ab \leq A_1 + A_2$ or $ab \leq \dfrac{a^p}{p} + \dfrac{b^q}{q}$. also geometrically, one can see that the equality holds if $a = b$.

Proof. **Proof of Theorem 3.3 part 2**
In the lemma, take $a = \dfrac{|f(t)|}{\|f\|_p}$ and $b = \dfrac{|g(t)|}{\|g\|_q}$ and integrate to obtain

$$\int_X |f(t)||g(t)|d\mu(t) \leq \|f\|_p\|g\|_q$$

Proof of Theorem 3.3 part 1
Note $|f+g|^p = |f+g||f+g|^{p-1}$. As $\dfrac{1}{p} + \dfrac{1}{q} = 1$, we can show that $|f+g|^{p-1} \in L_q$, indeed since $p = q(p-1)$, we get $\int_X |f+g|^{q(p-1)}d\mu = \int_X |f+g|^p d\mu < \infty$
Part 1 follows from part 2.

$$\int_X |f+g|^p d\mu = \int_X |f+g||f+g|^{p-1}d\mu \leq \int_X |f||f+g|^{p-1}d\mu + \int_X |g||f+g|^{p-1}d\mu < \infty$$

Using part 2 in these last integrals, we conclude that

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p$$

□

Definition 3.3. *Let X be a vector space, we define an inner product in X as a function $<,>: X \times X \longrightarrow \mathbb{R}$ or $\mathbb{C}$. That satisfy the following properties for all $x, y, z \in X, , \mu \in \mathbb{R}$ or $\mathbb{C}$.*

1. $< x, x > \geq 0, < x, x >= 0$ *if and only if* $x = 0$.
2. $< x, y >= \overline{< y, x >}$ *where the bar means the conjugate of complex numbers.*
3. $< \lambda x + \mu y, z >= \lambda < x, z > + \mu < y, z >$.

The vector space X endowed with a inner product is called an **inner product space**.
Example 3.5. Let $X = \mathbb{R}^n$, define $<,>$ in $\mathbb{R}^n$ by for $x = (x_1, \cdots, x_n), y = (y_1, \cdots, x_n)$,

$$< x, y >= x_1 y_1 + \cdots + x_n y_n = \sum_{i=1}^{n} x_i y_i$$

One can show that $<,>$ is an inner product in $X = \mathbb{R}^n$. in case that we take $X = \mathbb{C}^n, u = (u_1, \cdots, u_n), v = (v_1, , v_n)$

$$< u, v >= u_1 \bar{v}_1 + \cdots + u_n \bar{v}_n = \sum_{i=1}^{n} u_i \bar{v}_i$$

.

Example 3.6. Let $X = l_2$ (Example 2.14)
$x = (x_1, \cdots, x_n, \cdots), y = (y_1, \cdots, y_n, \cdots) \in l_2$.

$$< x, y >= \sum_{i=1}^{\infty} x_i \bar{y}_i \tag{3.1}$$

is an inner product. The bar means the complex conjugation in $\mathbb{C}$, the complex numbers. If we use the real numbers $\mathbb{R}$, we don't need it.

Remark 3.18. *The convergence of the series 3.1 follows as a consequence of the Cauchy-Schwarz inequality below.*

Example 3.7. Let $X = C[a,b]$. Define $<,>$ as

$$< f, g >= \int_a^b f(x)g(x)dx$$

is an inner product.

Example 3.8. Let $X = L_2(S, \mathcal{F}, \mu)$. Define $<,>$ as

$$< f, g >= \int_X f(t)g(t)d\mu(t)$$

is an inner product.

Remark 3.19. *The next results are very important for spaces endowed with inner product.*

Theorem 3.4. *(Cauchy-Schwarz inequality) Let X be an inner product space and $x, y \in X$. Then*

$$| < x, y > |^2 \leq \quad < x, x > \cdot < y, y >$$

Equality holds of and only if x abd y are linearly dependent.

Proof. Let's consider the inner product of the vector $\lambda x - \mu y$ by itself where $x, y \in \mathbb{R}^n$, λ, μ are scalars.

$$< \lambda x - \mu y, \lambda x - \mu y >= |\lambda|^2 < x, x > +|\mu|^2 < y, y > -\overline{\lambda}\mu < x, y > -\lambda\overline{\mu} < y, x >\geq 0$$

Since $< \lambda x - \mu y, \lambda x - \mu y >\geq 0$ for all λ, μ scalars, let's take $\lambda =< x, y >, \mu =< x, x >$ then

$$| < x, y > |^2 < x, x > +| < x, x > |^2 < y, y > -2 < x, x > | < x, y > |^2 \geq 0$$

which implies

$$| < x, y > |^2 \leq \quad < x, x > \cdot < y, y >$$

It is trivial to show that the equality holds if and only if x, y are linearly dependent. □

Theorem 3.5. *The functions $\| \cdot \|_X : X \longrightarrow \mathbb{R}$ defined by $\|x\|_X = \sqrt{< x, x >}$ is a norm in X.*

Theorem 3.6. *The inner product* $<,>: X \times X \longrightarrow \mathbb{R}$ *or* $\mathbb{C}$ *is a continuous funtion in* $X \times X$.

Remark 3.20. *The proof of 3.5 and 3.6 are immediate consequence of the propeties and results on inner product.*

Theorem 3.7 (Parallelogram Law). *Let* X *be an inner product space; then for any* $x, y \in X$, *we have:*

$$\|x+y\|_X^2 + \|x-y\|_X^2 = 2(\|x\|_X^2 + \|y\|_X^2)$$

Proof. To see this equality, just expand the left-hand side and take into consideration that $\|x\|_X^2 =< x, x >$

□

Remark 3.21. *Not every norm comes from an inner product, for example from the 3 norms we defined in the example 2.10 for* $\mathbb{R}^n$, $\|\cdot\|_1$, $\|\cdot\|_\infty$ *and* $\|\cdot\|_2$, *just* $\|\cdot\|_2$ *comes from an inner product. Indeed,* $x = (x_1, \cdots, x_n)$,

$$\|x\|_2 = \sqrt{x_1^2 + \cdots + x_n^2} = \sqrt{< x, x >}$$

.

Remark 3.22. *For the spaces* $L_p, 1 \leq p \leq \infty$ *defined in the example 3.4, just for* $p = 2$ *the norm* $\|\cdot\|_p$ *comes from an inner product. In fact*

$$\|f\|_2 = \left(\int_X |f(t)^2 d\mu(t)| \right)^{1/2} = \left(\int_X f(t) f(t) d\mu(t) \right)^{1/2} = \sqrt{< f, f >}$$

Remark 3.23. *A vector space* X *endowed with an inner product is a normed vector space where the nom is given by*

$$\|x\|_X = \sqrt{< x, x >}.$$

Remark 3.24. *It is important to note that if the norm in* X *comes from an inner product space, then this norm have to satisfy the parallelogram Law (Theoem 3.7).*

Definition 3.4. *An inner product* X *which is complete is called a Hilbert space.*

Example 3.9. Let $X = C[-1,1]$ and $< f, g >= \int_{-1}^{1} f(t)g(t)dt$.
$C[-1,1]$ is an inner product space, but is not a Hilbert space, since it is not complete(Example 2.7)

Example 3.10. Let $X = l_2$ and $< x, y >= \sum_{n=1}^{\infty} x_n y_n$ where $x = (x_1, \cdots, x_n, \cdots), y = (y_1, \cdots, y_n, \cdots)$. l_2 is a Hilbert space.

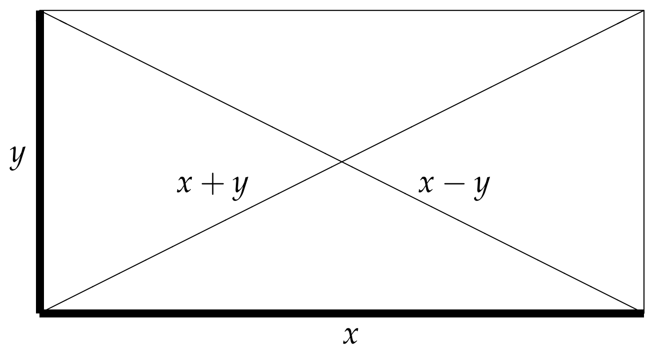
Example 3.10. Let $X = L_1[0,1]$ and $\|f\|_1 = \int_0^1 |f(t)|dt$.
We can show that this norm $\|f\|_1$ does not come from an inner product. To see that, take $f(t) = \chi_A(t), g(t) = \chi_B(t)$ where $A = [0, 1/2)$ and $B = [1/2, 1]$.
Then $\|f+g\|_1^2 + \|f-g\|_1^2 \neq 2(\|f\|_1^2 + \|g\|_1^2)$.
In fact $\|\chi_A + \chi_B\|_1 = \int_A \chi_A(t)dt + \int_B \chi_B(t)dt = \int_0^{1/2} dt + \int_{1/2}^1 dt = 1$.
Likewise $\|\chi_A - \chi_B\|_1 = 1$ whereas $\|\chi_A\|_1 = 1/2$ and $\|\chi_B\|_1 = 1/2$.
Therefore we get: $2 = \|\chi_A + \chi_B\|_1^2 + \|\chi_A - \chi_B\|_1^2 \neq 2(\|\chi_A\|_1^2 + \|\chi_B\|_1^2) = 2(1/4 + 1/4) = 1$

Definition 3.5. *Let X be an inner product space. We say that the vector space x is perpendicular to a vector $y \in X$ and denote by $x \perp y$ if $< x, y >= 0$*

Theorem 3.8 (The Pythagorean Theorem for an inner product space)**.** *Let X be an inner product space, then*

$$\textit{If} \quad x \perp y \quad \textit{then} \quad \|x+y\|_X^2 = \|x\|_X^2 + \|y\|_X^2$$

Geometrically:



y
$x + y$
$x - y$
x

CHAPTER 4

LINEAR TRANSFORMATIONS AND CHARACTERIZATION OF BOUNDED LINEAR FUNCTIONALS

Definition 4.1. *Let X, Y be vector spaces over a field F. A function $T : X \longrightarrow Y$ is said to be a linear transformation if*

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y), \quad \forall x, y \in X, \alpha, \beta \in F$$

.

Remark 4.1. *Some author use the terminology of Linear Operator rather than Linear Transformation. Moreover the condition in the definition split into two, namely:*

1. $T(x + y) = T(x) + T(y), \quad \forall x, y \in X$
2. $T(\alpha x) = \alpha T(x), \quad \forall x \in X, \forall \alpha \in F.$

Definition 4.2. *If in the definition 4.1, the space Y is replaced by the field F, then the linear transformation is usually called linear functional.*

Example 4.1. Differentiation, Integration are examples of linear transformation

i) P=the vector space of polynomials:

$$D : P \longrightarrow P, p \longmapsto Dp = \frac{dp}{dx}$$

ii) $C[a, b]$ = the vector space of the continuous functions in $[a, b]$.

$$I : C[a, b] \longrightarrow \mathbb{R}, f \longmapsto I(f) = \int_a^b f(t)dt$$

Example 4.2.

i) $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad T(x,y) = (\alpha x, \alpha y). \quad x,y \in \mathbb{R}^2, \alpha \in \mathbb{R}$. (Multiplication by a scalar)

ii) $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad T(x,y) = (y,x)$. T is reflection through the diagonal line.

iii) $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad T(x,y) = (x,0)$. T is the projection of $\mathbb{R}^2$ onto the x-axis.

(iv) $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad T(x,y) = (0,y)$. T is the projection of $\mathbb{R}^2$ onto the y-axis.

These are examples of linear transformations.

Example 4.3. $T : l_2 \longrightarrow l_2, \quad x = (x_1, \cdots, x_n, \cdots) \quad T(x) = (0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \cdots, \frac{x_n}{n}, \cdots)$. T is linear.

Note: Some elementary properties of linear transformations are given the next theorem.

Theorem 4.1. *Let X, Y be vector spaces over a field F and $T : X \longrightarrow Y$ be a linear Transformation. Then:*

i) $T(0) = 0$

ii) The range of T, $R(T) = \{y \in Y : T(x) = y \quad \text{for some} \quad x \in X\}$ is subspace of Y.

iii) T is injective if ans only $T(x) = 0 \Longrightarrow x = 0$.

iv) If T is injective, then T^{-1} exists in $R(T)$ and $T^{-1} : R(T) \longrightarrow X$ is also linear.

Definition 4.3. *Let X, Y be normed vector spaces and $T : X \longrightarrow Y$ a linear transformation. We say, that T is a bounded linear transformation if there is an absolute constant $M > 0$ such that $\|Tx\|_Y \leq M\|x\|_X$*

Definition 4.4. *Let X, Y be normed vector spaces and $T : X \longrightarrow Y$ a linear transformation. We say that T is continuous is X if for any sequence (x_n) in X such that*

$$x_n \longrightarrow x \quad in \quad X \quad then \quad T(x_n) \longrightarrow T(x) \quad in \quad Y$$

Theorem 4.2. *$T : X \longrightarrow Y$ is continuous in $x \in X$ if and only T is continuous at zero.*

Remark 4.2. *The proof of Theorem 4.2 is a direct application of the properties of linearity of T.*

Theorem 4.3. *A linear transformation $T : X \longrightarrow Y$ is bounded if and only if T is continuous.*

Proof. $\Longrightarrow$) Let (x_n) in X be a convergent sequence, say $x_n \longrightarrow x$ as $n \longrightarrow \infty$, that is $\|x_n - x\|_X \longrightarrow 0$ as $n \longrightarrow \infty$. Now since T is bounded,we have:
$\|T(x_n - x)\|_Y \leq M\|x_n - x\|_X$. As $\|x_n - x\|_X \longrightarrow 0$ as $n \longrightarrow \infty$, so is $\|T(x_n - x)\|_Y \longrightarrow 0$ as $n \longrightarrow \infty$.
$\Longleftarrow$). The proof of the other direction will be made by contradiction, that is assume that there is no $M > 0$ so that $\|Tx\|_Y \leq M\|x\|_X$. Then for any positive integer n, there is $x_n \in X, x_n \neq 0$ so that $\|T(x_n)\|_Y > n\|x_n\|_X$, this implies that $\dfrac{\|T(x_n)\|_Y}{n\|x_n\|_X} > 1$.
Define the sequence $u_n = \dfrac{x_n}{n\|x_n\|}$ so that $\|u_n - 0\|_X = \left\|\dfrac{x_n}{n\|x_n\|_X}\right\|_X = \dfrac{1}{n} \longrightarrow 0$ as $n \longrightarrow \infty$,i.e. $u_n \longrightarrow 0$ in X.
On the other hand, $\|Tu_n - 0\|_Y = \|Tu_n\|_Y = \left\|\dfrac{T(x_n)}{n\|x_n\|_X}\right\|_Y = \dfrac{\|T(x_n)\|_Y}{n\|x_n\|_X} > 1$ so that $Tu_n \nrightarrow 0$ contradiction.

$\square$

Remark 4.3. *The idea of the proof by contradiction is to negate the hypothesis that T is not bounded and to construct a sequence (u_n) in X that converges to $0 \in X$, but the sequence $T(u_n)$ does not converges to $T(0) = 0$.*

Remark 4.4. *If $T : X \longrightarrow Y$ is a bounded linear transformation, then we have the inequality $\|Tx\|_Y \leq M\|x\|_X$ for some absolute constant $M > 0$ and for all $x \in X$. it is obvious the constant M in this inequality is not unique. Indeed if $\|Tx\|_Y \leq 7\|x\|_X$, then $\|Tx\|_Y \leq 8\|x\|_X \leq \cdots$*
However, in general M is bounded from below. in fact 0 is a lower bound of M. This will tell us that $Inf\{M : \|Tx\|_Y \leq \|x\|_X\}$ exists for all $x \in X$. This observation suggests the next definition.

Definition 4.5. *Let $T : X \longrightarrow Y$ be a linear transformation defined in the normed spaces X and Y, then we define the norm $\|T\|$ as*

$$\|T\| = Inf\{M > 0 : \|Tx\|_Y \leq M\|x\|_X, \forall x \in X\}$$

.

Remark 4.5. *The definition 4.5 immediatly implies that* $|Tx\|_Y \leq \|T\|\|x\|_X$ *and that for every* $\epsilon > 0$*, there exists an* $x_\epsilon \in X, x_\epsilon \neq 0$ *so that* $\|Tx_\epsilon\|_Y > (\|T\| - \epsilon)\|x_\epsilon\|_X$

Remark 4.6. *If* $\|Tx\|_Y \leq M\|x\|_X$*, then we can write* $\dfrac{\|Tx\|_Y}{\|\|_X} \leq M$*; Therefore we can also define* $\|T\|$ *by*

$$\|T\| = \underset{x\neq 0}{Sup}\frac{\|Tx\|_Y}{\|x\|_X}$$

Theorem 4.4. *Let* $T : X \longrightarrow Y$ *be a bounded linear transformation* X, Y *normed spaces, then we have* $\|T\| = \underset{\|x\|_X\leq 1}{Sup}\|T(x)\|_Y = \underset{\|x\|_X=1}{Sup}\|T(x)\|_Y = \underset{x\neq 0}{Sup}\dfrac{\|T(x)\|_Y}{\|x\|_Y}$

Proof. Let $\|T\|' = \underset{\|x\|_X=1}{\text{Sup}}\|T(x)\|_Y, \|T\|'' = \underset{\|x\|_X\leq 1}{\text{Sup}}\|T(x)\|_Y$ and $\|T\|''' = \underset{x\neq 0}{\text{Sup}}\dfrac{\|T(x)\|_Y}{\|x\|_Y}$.
Define the set $B = \{x \in X; \|x\|_X \leq 1\}$; then $\partial B = \{x \in X; \|x\|_X = 1\}$ so that $\partial B \subset B \Longrightarrow \underset{x\in\partial B}{\text{Sup}}\|Tx\|_Y \leq \underset{x\in B}{\text{Sup}}\|Tx\|_Y$, that is $\|T\|' \leq$
$|T\|''$ **(1)**
Note that $\dfrac{x}{\|x\|_X} \in \partial B, x \neq 0$. Since $\left\|\dfrac{x}{\|x\|_X}\right\|_X = 1$, then

$$\underset{xneq0}{\text{Sup}}\frac{\|T(x)\|_Y}{\|x\|_X} = \underset{x\neq 0}{\text{Sup}}\left\|T(\frac{x}{\|x\|_X})\right\|_Y \leq \underset{\|x\|_X=1}{\text{sup}}\|T(x)\|_Y,$$

so that

$$\|T\|''' \leq \|T\|' \quad \textbf{(2)}.$$

Finally, if $\|x\|_X \leq 1, x \neq 0$, then $\|T(x)\|_Y \leq \dfrac{1}{\|x\|_X} \cdot \|T(x)\|_Y$. Since $\dfrac{1}{\|x\|_X} \geq 1$, therefore

$$\underset{\|x\|_X\leq 1}{\text{Sup}}\|T(x)\|_Y \leq \underset{x\neq 0}{\text{Sup}}\frac{\|T(x)\|_Y}{\|x\|_X},$$

that is

$$\|T\|'' \leq \|T\|''' \quad \textbf{(3)}.$$

Putting **(1)**, **(2)** and **(3)** together, we get $\|T\|' \leq \|T\|'' \leq \|T\|''' \leq \|T\|'$ which implies that $\|T\|' = \|T\|'' = \|T\|'''$ □

Example 4.4. Let $T : l_2 \longrightarrow l_2$ be defined by $T(x_1, \cdots x_n, \cdots) = (0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \cdots, \frac{x_n}{n}, \cdots)$.

$$\|T(x)\|_{l_2} = \left(\sum_{n=1}^{\infty} \left| \frac{x_n}{n} \right|^2 \right)^{\frac{1}{2}} \leq \left(\sum_{n=1}^{\infty} \frac{1}{n^2} |x_n|^2 \right)^{\frac{1}{2}} \leq \left(\sum_{n=1}^{\infty} |x_n|^2 \right)^{\frac{1}{2}}$$

that is $\|Tx\|_{l_2} \leq \|x\|_{l_2}$.
So T is a bounded linear transformation. Moreover, $\|T\| \leq 1$.
Now if we take $e_1 = (1, 0, \cdots, 0, \cdots)$, then $T(e_1) = (0, 1, 0, \cdots, 0, \cdots)$ and $\|T(e_1)\|_{l_2} = 1$.
Consequenly $\|T\| = \underset{\|x\|_{l_2}}{Sup} \|T(x)\|_Y \geq 1$. So putting together $\|T\| \leq 1$ and $\|T\| \geq 1$, we get $\|T\| = 1$

Example 4.5. Let $T : C[a, b] \longrightarrow \mathbb{R}$ be defined by $Tf = \int_a^b f(t)dt, C[a, b]$ endowed with the Sup norm $\|f\|_\infty = \underset{a \leq t \leq b}{Sup} |f(t)|$.
T is a bounded linear transformation, moreover

$$|Tf| = \left| \int_a^b f(t)dt \right| \leq \int_a^b |f(t)dt| \leq \|f\|_\infty \int_a^b dt = (b-a)\|f\|_\infty$$

that is,

$$|Tf| \leq (b-a)\|f\|_\infty.$$

Therefore by definition of $\| \cdot \|$, we get $\|T\| \leq (b-a)$. On the other hand, if we take $g(t) = 1$ for $a \leq t \leq b$,$\|g\|_\infty = 1$ and $|Tf| = \int_a^b 1dt = b - a$ so that

$$\|T\| = \underset{\|f\|_\infty = 1}{\text{Sup}} |Tf| \geq b - a$$

consequently $\|T\| = b - a$.

Example 4.6. $P[0, 1]$=Set of polynomials defined in $[0, 1]$.

$$\begin{aligned} D : P[0,1] &\longrightarrow P[0,1] \\ p &\longmapsto Dp = \frac{dp}{dx} = p' \end{aligned}$$

Define the following norms in $P[0,1]$ by

$$\|p\|_\infty = \sup_{0\le t\le 1} |p(t)| \quad \|p\|'' = \max\{\|p\|_\infty, \|p'\|_\infty\}$$

i)

$$\begin{aligned} D : (P[0,1], \|.\|_\infty) &\longrightarrow (P[0,1], \|.\|_\infty) \\ p &\longmapsto Dp = \frac{dp}{dx} = p' \end{aligned}$$

Note that $D(t^n) = nt^{n-1}$ where $p(t) = t^n$, $\|Dp\|_\infty = \|p'\|_\infty = n$ and $\|p\|_\infty = 1$. Therefore D is not bounded

ii)

$$\begin{aligned} D : (P[0,1], \|.\|'') &\longrightarrow (P[0,1], \|.\|_\infty) \\ p &\longmapsto Dp = p' \end{aligned}$$

$\|Dp\|_\infty = \|p'\|_\infty \le \max\{\|p\|_\infty, \|p'\|_\infty\} = \|p\|''$. Therefore $\|Dp\|_\infty = \|p\|''$. Hence D is bounded.

Note that the boundedness of a linear transformation depends on the norms in the spaces.

Definition 4.6. *Let X and Y be normed spaces and denote $B(X,Y)$ the set of all bounded linear transformations from X to Y; moreover define the operations of addition and scalar multiplication by $(T+L)(x) = T(x) + L(x)$, $(\alpha T)(x) = \alpha T(x)$, for all $T, L \in B(X,Y)$ and α a scalar. Then $B(X,Y)$ endowed with these operations becomes a vector space; moreover if we define*

$$\|T\| = \sup_{\|x\|_X=1} \|Tx\|_Y,$$

then $B(X,Y)$ is a normed vector space.

Theorem 4.5. *Let X and Y be normed vector spaces. Then $B(X,Y)$ is a Banach space if Y is a Banach space.*

Remark 4.7. *Given any two normed spaces, X and Y, it is almost impossible in general to characterize all the bounded linear transformations $T : X \longrightarrow Y$; in other words, to char-*

acterize explicitly the normed space $B(X,Y)$. However in several cases, this is possible. For example for $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$. Also when Y is a scalar field $\mathbb{R}$ or $\mathbb{C}$ and for some X.

Theorem 4.6. *Let $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$, $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ a bounded linear transformation, then there is a $m \times n$ matrix A so that $Tx = Ax$. Conversely if a $m \times n$ matrix A is given, then T defined by $Tx = Ax$ is bounded linear transformation. This means if we denote the set of all $m \times n$ matrix by $M_{m\times n}$, the transformation ψ defined by*

$$\begin{aligned} \psi : B(\mathbb{R}^n, \mathbb{R}^m) &\longrightarrow M_{m\times n} \\ T &\longmapsto \psi(T) = A \end{aligned}$$

is a bijection. Therefore $B(\mathbb{R}^n, \mathbb{R}^m)$ is identified as the vector spaces of the matrices $M_{m\times n}$, that is, $B(\mathbb{R}^n, \mathbb{R}^m) \cong M_{m\times n}$.

Remark 4.8. *In the particular case when $Y = \mathbb{R}$ or $\mathbb{C}$, $B(X,Y)$ is called the dual space of X and it is denoted by X^*. It is a very difficult and important problem in analysis, to characterize the dual space of a given Banach space X. From a corollary of the Hahn-Banach extension theorem, we have that Banach space $X \neq \{0\}$, then $X^* \neq \{0\}$.*

Remark 4.9. *If we take $X = \mathbb{R}^n$ then the Riesz's representation theorem claims that $B(\mathbb{R}^n, \mathbb{R}) = (\mathbb{R}^n)^*$ is equivalent as a Banach space to $\mathbb{R}^n$, that is, $(\mathbb{R}^n)^* \cong \mathbb{R}^n$. Indeed we have the following result, well-known as the Riesz's Representation Theorem.*

Theorem 4.7. *Let fix $y \in \mathbb{R}^n$ and define $\psi_y(x) = \langle x, y\rangle$ where $\langle , \rangle$ is the inner product in $\mathbb{R}^n$. Then ψ_y is a bounded linear functional. Conversely if $\psi \in (\mathbb{R}^n)^*$, there is a unique $y \in \mathbb{R}^n$ so that $\psi = \psi_y$; moreover*

$$\|\psi\| = \|y\|_{\mathbb{R}^n}.$$

Proof. To see that ψ_y is linear, all we need is to use the properties of the inner product. For the boundedness, we need Theorem 3.4(The Cauchy-Shwarz inequality), that is

$$|\psi_y(x)| = |\langle x, y\rangle| \leq \|y\|_{\mathbb{R}^n} \|x\|_{\mathbb{R}^n},$$

moreover

$$\|\psi_y\| \leq \|y\|_{\mathbb{R}^n}. \tag{4.1}$$

On the other hand, if $\psi(\mathbb{R}^n)^*$, $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ and $\{e_i\}_{i=1}^n$, the canonical basis for $\mathbb{R}^n$ *i.e.* $e_1 = (1, 0, ..., 0)$, $e_2 = (0, 1, 0, ..., 0)$,...,$e_n = (0, 0, ..., 0, 1)$. Then $x = \sum_{i=1}^{n} x_i e_i$ and $\psi(x) = \sum_{i=1}^{n} x_i \psi(e_i)$, therefore if we take $y = \psi(e_1, ..., \psi(e_n))$. We get that $\psi = \psi_y$, that is, ψ is an inner product. If we evaluate $\psi(y)$, then

$$\psi(y) = \langle y, y \rangle = \|y\|^2_{\mathbb{R}^n}, \ so \ \psi_y(y) = \|y\|_{\mathbb{R}^n} \|y\|_{\mathbb{R}^n},$$

consequently $\dfrac{\psi_y(y)}{\|y\|_{\mathbb{R}^n}} = \|y\|_{\mathbb{R}^n}$ and so

$$\|\psi_y\| = \sup_{y \neq 0} \frac{\psi_y(y)}{\|y\|_{\mathbb{R}^n}} \geq \|y\|_{\mathbb{R}^n}. \tag{4.2}$$

Putting together (4.1) and (4.2), we get $\|\psi_y\| = \|\psi\| = \|y\|_{\mathbb{R}^n}$. One way to see what we have done is: Define

$$\begin{aligned} T : \mathbb{R}^n &\longrightarrow (\mathbb{R}^n)^* \\ y &\longmapsto T(y) = \psi_y, \ \ where \ \psi_y(x) = \langle x, y \rangle. \end{aligned}$$

T is an isometry, that is T is bijective and

$$\|T(y)\|_{(\mathbb{R}^n)^*} = \|y\|_{\mathbb{R}^n}.$$

$\square$

Theorem 4.8. *(Riesz's Representation Theorem for Hilbert space) Let H be a Hilbert space and $\psi \in H^*$. Then*

i) There is a unique $y \in H$ so that $\psi(x) = \langle x, y \rangle$

ii) $\|\psi\| = \|y\|_H$.

Remark 4.10. *The Cauchy-Shwarz inequality for inner product space (Theorem 3.4), claims that if we fix $y \in H$ and define $\psi_y(x) = \langle x, y \rangle$, then ψ_y is a bounded linear functional, therefore the Riesz's Representation Theorem, claims that only bounded linear functional on H are the inner product. Moreover, if we define $T : H \longrightarrow H^*$ by $T(y) = \psi_y$ where $\psi_y(x) = \langle x, y \rangle$, is an isometry and so $H^* \cong H$.*

Proof. (**Proof of Theorem 4.8**) Let $\psi \in H^*$. If $\psi(x) = 0$, $\forall\, x \in H$, then $y = 0 \in H$ is sufficient. Let's assume that $\psi \neq 0$, define $M = \{x \in H : \ \psi(x) = 0\}$. As ψ is bounded (continuous), it is easy to show that M is a proper closed subspace of H. Therefore the theorem of orthogonal complement, implies that there is $0 \neq y_0 \in H \setminus M$ so that $y_0 \perp M$ $(*)$ *i.e.* $\langle x, y_0 \rangle = 0$, $\forall\, x \in M$.

Idea: Find a scalar α so that $y = \alpha y_0$ is the desired element in H satisfying our requirement. Note for $x \in M$, it does not matter what α is. In fact if $x \in M$, $\psi(x) = 0$. On the other hand $\langle x, y \rangle = \langle x, \alpha y_0 \rangle = \alpha \langle x, y_0 \rangle = \alpha.0 = 0$, therefore $\psi(x) = \langle x, y \rangle$, $\forall\, x \in M$. Let find α so that $\psi(x) = \langle x, y \rangle$ for $y = \alpha y_0$, it is true for $x = y_0$, that is,

$$\psi(y_0) = \langle y_0, \alpha y_0 \rangle = \alpha \langle y_0, y_0 \rangle = \|y_0\|_H^2,$$

so that $\alpha = \psi(y_0)\|y_0\|_H^2$. Now get any $x \in H$ and let's find a scalar β so that $x - \beta y_0 \in M$. In fact $\psi(x - \beta y_0) = 0 \Rightarrow \psi(x) - \beta\psi(y_0) = 0$ and so $\beta = \dfrac{\psi(x)}{\psi(y_0)}$. Indeed for this β, $x - \beta y_0 \in M$. Let x be any element in H, then we can write x as $x = x - \dfrac{\psi(x)}{\psi(y_0)} y_0 + \dfrac{\psi(x)}{\psi(y_0)} y_0$, now take $y = \dfrac{\psi(y_0)}{\|y_0\|_H^2} y_0$ then $\langle x, y \rangle = \psi(x)$. Indeed

$$\begin{aligned}
\langle x, y \rangle &= \langle x - \frac{\psi(x)}{\psi(y_0)} y_0 + \frac{\psi(x)}{\psi(y_0)} y_0, \frac{\psi(y_0)}{\|y_0\|_H^2} \rangle \\
&= \langle x - \frac{\psi(x)}{\psi(y_0)} y_0, \frac{\psi(y_0)}{\|y_0\|_H^2} \rangle + \langle \frac{\psi(x)}{\psi(y_0)} y_0, \frac{\psi(y_0)}{\|y_0\|_H^2} \rangle \\
&= 0 + \frac{\psi(x)}{\psi(y_0)} \frac{\psi(y_0)}{\|y_0\|_H^2} \langle y_0, y_0 \rangle \\
&= \psi(x).
\end{aligned}$$

that is, $\psi(x) = \langle x, y \rangle$ for $y = \dfrac{\psi(y_0)}{\|y_0\|_H^2} y_0$ and so $\psi = \psi_y$. This prove part i). To see part ii), notice $|\psi(x)| = |\langle x, y \rangle| \leq \|x\|_H \|y\|_H$. Therefore

$$\|\psi\| \leq \|y\|_H. \tag{4.3}$$

On the other hand, $\psi(y) = \langle x, y \rangle = \|y\|_H^2 = \|y\|_H \|y\|_H$ and so

$$|\psi(y)| = \left| \psi\left(\frac{y}{\|y\|_H}\right) \right| = \|y\|_H$$

which implies that $\sup_{\|x\|_H=1} |\psi(x)| \geq \|y\|_H$ since $\left\| \frac{y}{\|y\|_H} \right\| = 1$ and so

$$\|\psi\| \geq \|y\|_H. \tag{4.4}$$

Putting (4.3) and (4.4) together, we get $\|\psi\| = \|y\|_H$. □

Remark 4.11. *$y \in H$ in the theorem is unique.*

Remark 4.12. *In $(*)$ above, we said that there is $y_0 \in H \setminus M$ so that $y_0 \perp M$. If such y_0 does not exist, then $M = \{x \in H : \ x \perp M\} = \{0\}$, this implies $M^{\perp\perp} = \{x \in H : \ x \perp M^{\perp}\} = \{0\}^{\perp} = H$. But $M^{\perp\perp} = M$, so $M = H$ which is a contradiction since M is a proper subspace of H.*

Theorem 4.9. *If M is a proper closed subspace of H, then $M^{\perp\perp} = M$.*

Example 4.7. Define $\psi_i : l_2 \longrightarrow \mathbb{R}$ by $\psi_i(x) = \psi_i(x_1, ..., x_i, ..., x_n, ...) = x_i$. ψ_i is a bounded linear functional. Indeed,

$$|\psi_i(x)| = |x_i| = \left(|x_i|^2\right)^{1/2} \leq \left(\sum_{n=1}^{\infty} |x_n|\right)^{1/2} = \|x\|_2$$

i.e. $|\psi_i(x)| \leq \|x\|_2 \ \forall \ x \in l_2$, $i = 1, 2, 3, ...$ Since ψ_i is a bounded linear functional in the Hilbert space l_2, let's find $y \in l_2$ as in the Riesz's Representation Theorem so that $\psi_i(x) = \langle x, y \rangle$. Let $y = (y_1, ..., y_n, ...)$ and $x = (x_1, ..., x_n, ...)$, we want to find y so that $\langle x, y \rangle = x_i$, in order to get this, set $y_i = 1$, $y_n = 0$ if $n \neq i$. So taking y as $y = e_i = (0, 0, ..., 0, 1, 0, ...)$, we get $\langle x, e_i \rangle = x_i$, that is, $\psi_i(x) = \langle x, e_i \rangle$.

Example 4.8. Define $\psi : L_2[0, 2\pi] \longrightarrow \mathbb{R}$ by $\psi(f) = \int_0^{2\pi} f(t)dt \ \forall \ f \in L_2[0, 2\pi]$. Using properties of integrals, we show that ψ is linear; moreover the *Hölder*'s inequality theorem 3.3 part 2, above imply that

$$|\psi(f)| = \left| \int_0^{2\pi} f(t)dt \right| \leq \int_0^{2\pi} 1.|f(t)|dt \leq \left(\int_0^{2\pi} 1^2 dt\right)^{1/2} \left(\int_0^{2\pi} |f(t)|^2 dt\right)^{1/2}$$

which implies that $|\psi(f)| \leq \sqrt{2\pi}\|f\|_2$, so ψ is also bounded. Let's find a function

$g : L_2[0, 2\pi] \longrightarrow \mathbb{R}$ so that

$$\psi(f) = \langle f, g\rangle = \int_0^{2\pi} f(t)g(t)dt = \int_0^{2\pi} f(t)dt.$$

We can easily see that $g(t) = 1$ on $[0, 2\pi]$ and so $\psi(f) = \langle f, g\rangle$.

Theorem 4.10. *(Riesz's Representation Theorem for L_p for $1 < p < \infty$)*
$\varphi \in (L_p)^*$ if and only if there is a unique $g \in L_q$, $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p < \infty$ so that $\varphi = \varphi_g$ where $\varphi_g(f) = \int_X f(t)g(t)d\mu(t)$, moreover $\|\varphi\| = \|\varphi_g\| = \|g\|_q$.

Proof. We are going to present the steps of the proof, but without much details.
Let $g \in L_q$, $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p < \infty$, define $\varphi_g(f) = \int_X f(t)g(t)d\mu(t)$, then the *Hölder*'s inequality Theorem 3.3 above imply that $|\varphi_g(f)| \leq \|g\|_q\|f\|_p$. Then φ_g is bounded and $\|\varphi_g\| \leq \|g\|_q$. On the other hand, if we get $\varphi \in (L_p)^*$ and define the set function λ by $\lambda(A) = \varphi(\chi_A)$, where χ_A is the characteristic function of the measurable set A. Then the boundedness of φ implies that

$$|\lambda(A)| = |\varphi(\chi_A)| \leq \|\varphi\|\|\chi_A\|_p.$$

As $\|\chi_A\|_p = (\mu(A))^{1/p}$, we have

$$|\lambda(A)| \leq \|\varphi\|(\mu(A))^{1/p}$$

Consequently λ is absolutely with respect to μ *i.e.* $\lambda \ll \mu$. Therefore the Radon-Nikodym Theorem, implies that there is an integrable function g so that

$$\varphi(\chi_A) = \lambda(A) = \int_A g(t)d\mu(t) = \int_X \chi_A(t)g(t)d\mu(t).$$

One can show that this can be extended to all $f \in L_p$, and so we get
$\varphi(f) = \int_X f(t)g(t)d\mu(t)$. Moreover, we can show that $g \in L_q$ and $\|\varphi\| = \|g\|_q$. □

Remark 4.13. *The Riesz's Representation Theorem tell us that $B(L_p, \mathbb{R}) \cong L_q$ for $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p < \infty$, since one can show that the linear transformation $T : L_q \longrightarrow (L_p)^*$ defined by $T(g) = \varphi_g$ where $\varphi_g(f) = \int_X f(t)g(t)d\mu(t)$ is an isometry.*

Example 4.9. $c_0 = \{x = (x_n) : \lim_{n\to\infty} x_n = 0\}$, $\|x\|_{c_0} = \sup_{n\geq 1} |x_n|$.
Define $\psi : c_0 \longrightarrow \mathbb{R}$ by $\psi(x) = \sum_{n=1}^{\infty} x_n y_n$, where $y = (y_n) \in l_1$.
We can show that $B(c_0, \mathbb{R}) \cong (c_0)^* \cong l_1$.

Example 4.10. Let $c = \{x = (x_n) : (x_n) \textit{ is a convergent sequence}\}$. $\|x\|_c = \sup_{n\geq 0} |x_n|$, define $\psi : c \longrightarrow \mathbb{R}$ by

$$\psi(x) = \sum_{n=0}^{\infty} x_n y_n + \lim_{n\to\infty} x_n (y_0 - \sum_{n=0}^{\infty} y_n)$$

or

$$\psi(x) = (\lim_{n\to\infty} x_n) y_0 + \sum_{k=1}^{\infty} y_k \lim_{n\to\infty} (x_n - x_k)$$

where $(y_n) \in l_1$. Again we can show that $B(c, \mathbb{R}) \cong c^* \cong l_1$.

Remark 4.14. *A little more details on the Example 4.9 above. Fix $y = (y_n)_{n\geq 1} \in l_1$, that is, $\sum_{n=1}^{\infty} |y_n| < \infty$. Define ψ by $\psi_y(x) = \sum_{n=1}^{\infty} x_n y_n$. It is immediate that ψ_y is linear; moreover $\|x\|_{c_0} = \sup_{n\geq 1} |x_n| \geq |x_n| \; \forall \; n \geq 1$, so that we have*

$$|\psi_y(x)| \leq \sum_{n=1}^{\infty} |x_n||y_n| \leq \|x\|_{c_0} \sum_{n=1}^{\infty} |y_n|$$

and so $|\psi_y(x)| \leq \|y\|_{l_1} \|x\|_{c_0}$. Therefore ψ_y is bounded. On the other hand, if $\psi \in (c_0)^*$, there is $y = (y_n) \in l_1$ so that $\psi = \psi_y$. Indeed, define $y = (\psi(e_n))$ where $e_1 = (1, 0, ..., 0, ...)$,...,$e_n = (0, ..., 0, 1, 0, ...)$,...
Claim: *$y = (y_n) \in l_1$, $y = (\psi(e_n))$. To see that take $x = \left(\frac{y_1}{|y_1|}, \frac{y_2}{|y_2|}, ..., \frac{y_n}{|y_n|}, 0, 0, ...\right)$*

$$x_n = \begin{cases} \frac{y_n}{|y_n|}, & 1 \leq n \leq k; \\ 0, & n > k. \end{cases} \tag{4.5}$$

It can be easily seen that $x = (x_n) \in c_0$. Moreover, for $x \in c_0$, since $x = \sum_{n=1}^{\infty} x_n e_n$, we have

$$\psi(x) = \sum_{n=1}^{\infty} x_n \psi(e_n) = \sum_{n=1}^{\infty} x_n y_n,$$

where $y_n = \psi(e_n)$. Consequently $\psi = \psi_y$. Also note that for $x = (x_n)$ in (4.5),

$$\sum_{n=1}^{k} |y_n| = \sum_{n=1}^{k} x_n y_n = \psi(x) = |\psi(x)| \leq \|\psi\| \|x\|_{c_0} = \|\psi\|,$$

since $\|x\|_{c_0} = 1$. Then $\sum_{n=1}^{k} |y_n| \leq \|\psi\|$ and so $\sum_{n=1}^{\infty} |y_n| < \infty$, which is $y = (y_n) \in l_1$.

Example 4.11. (De Souza's Representation Theorem or Duality for the De Souza's space)
Let B^1 be the De Souza's space in the example 2.24, which is, $f \in B^1$ if and only if $\sum_{n=0}^{\infty} c_n b_n$, where $b_n(t) = \frac{1}{|I_n|}[\chi_{L_n}(t) - \chi_{R_n}(t)]$, $b_0 = \frac{1}{2\pi}$, $I_n \subseteq [0, 2\pi]$ interval, $I_n = L_n \cup R_n$, L_n and R_n are the halves of I_n and $\|f\|_{B^1} = \operatorname{Inf} \sum_{n=1}^{\infty} |c_n|$, where the infimum is taken over all possible representations of f. Let's fix $g \in \Lambda_*$, the space of all continuous periodic functions of period 2π so that

$$|g(x+h) + g(x-h) - 2g(x)| \leq Mh.$$

This space Λ_* is well-known and called the Zygmund class of functions.
Define $\psi_g : B^1 \longrightarrow \mathbb{R}$ by $\psi_g(f) = \int_0^{2\pi} f(t) dg(t)$.
We can show that ψ_g is a linear functional, moreover $|\psi_g(f)| \leq \|g\|_{\Lambda_*} \|f\|_{B^1}$, where $\Lambda_*^1 = \{g' : \ g \in \Lambda_*\}$, g' is taken in the distribution sense. As a consequence ψ_g is a bounded linear functional. One can indeed show that $B(B^1, \mathbb{R}) \cong (B^1)^* \cong \Lambda_*^1$.

Example 4.12. Let c be the set of convergent sequences with $\|x\|_c = \sup_{n \geq 1} |x_n|$, $x = (x_n)$. A linear transformation $T : c \longrightarrow c$ can be represented in the form $Tx = Ax$

where A is an infinite matrix of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} & ... \\ a_{21} & a_{22} & ... & a_{2n} & ... \\ . & . & . & .... & \\ a_{m1} & a_{m2} & ... & a_{mn} & ... \\ . & . & ... & ... & \end{pmatrix} \tag{4.6}$$

If $x = (x_1, x_2, ..., x_n, ...)$, then

$$T(x) = (\sum_{i=1}^{\infty} a_{1i}x_i, \sum_{i=1}^{\infty} a_{2i}x_i, ..., \sum_{i=1}^{\infty} a_{mi}x_i, ...)$$

Now if one want to characterize the bounded linear transformations $T : c \longrightarrow c$, we just characterize matrices A above, and this established in a result by Silverman-Toeplitz, the following statement well-known as Toeplitz matrix.

Theorem 4.11. *(Silverman-Toeplitz' Theorem)*
Let $T : c \longrightarrow c$ be defined by

$$T(x) = (\sum_{i=1}^{\infty} a_{1i}x_i, \sum_{i=1}^{\infty} a_{2i}x_i, ..., \sum_{i=1}^{\infty} a_{mi}x_i, ...).$$

If $x = (x_n) \in c$ then $\sum_{i=1}^{n} a_{ni}x_i \in c$ and $\|Tx\|_c \leq M\|x\|_c$ if and only if the matrix A satisfies

i) $\sum_{i=1}^{\infty} |a_{ni}| \leq M,\ i = 1, 2, 3, ...$

ii) $\lim\limits_{n\to\infty} a_{ni} = 0$ *for each* i

iii) $\lim\limits_{n\to\infty} \sum_{i=0}^{\infty} a_{ni} = 1$

Remark 4.15. *The theorem of Silverman-Toeplitz, say that $B(c, c)$ are the matrices of the form (4.6) satisfying $i)$, $ii)$ and $iii)$.*

Remark 4.16. *If we define $a_{nk} = \begin{cases} \frac{1}{n}, & 1 \leq k \leq n; \\ 0, & k > n. \end{cases}$, then the matrix A in (4.6) can be* *written as*

$$A = \begin{pmatrix} 1 & 0 & ... & 0 & ... & \\ 1/2 & 1/2 & 0 & ... & 0 & ... \\ . & . & ... & ... & & \\ 1/n & 1/n & ... & 1/n & 0... & \\ . & . & ... & ... & & \end{pmatrix}$$

and the linear transformation T can be written as

$$T(x) = (x_1, \frac{x_1 + x_2}{2}, \frac{x_1 + x_2 + x_3}{3}, ..., \frac{x_1 + ... + x_n}{n}, ...)$$

this is well-known as the mean convergence of a sequence.

Example 4.13. Let $T : l_1 \longrightarrow l_1$, then for $x = (x_1, x_2, ..., x_n, ...)$, T can be written as

$$T(x) = (\sum_{i=1}^{\infty} a_{1i}x_i, \sum_{i=1}^{\infty} a_{2i}x_i, ..., \sum_{i=1}^{\infty} a_{ni}x_i, ...),$$

and so $Tx = Ax$, where A is an infinite matrix as in Example 4.12, so $T : l_1 \longrightarrow l_1$ is a bounded linear transformation if and only if $\sum_{n=1}^{\infty} |a_{nk}| < \infty$ (2). Therefore $B(l_1, l_1)$ are the infinite matrices satisfying (2).

Example 4.14. Let $T : l_\infty \longrightarrow l_\infty$, then for $x = (x_1, x_2, ..., x_n, ...) \in l_\infty$,

$$T(x) = (\sum_{i=1}^{\infty} a_{1i}x_i, \sum_{i=1}^{\infty} a_{2i}x_i, ..., \sum_{i=1}^{\infty} a_{ni}x_i, ...)$$

and $Tx = Ax$ where A is an infinite matrix. T is a bounded linear transformation if and only if $\sup_{n\geq 1} \sum_{k=1}^{\infty} |a_{nk}| < \infty$ (3) and so $B(l_\infty, l_\infty)$ are the infinite matrices satisfying (3).

Example 4.15. The **Hardy space** $H^p(\mathbb{D})$ is the space of the analytic functions F defined in the complex unit disc $\mathbb{D}$ so that

$$\|F\|_{H^p} = \sup_{0<r<1} \left(\int_0^{2\pi} |F(re^{i\theta})|^p d\theta \right)^{1/p} < \infty, \quad 0 < p < \infty.$$

For $1 < p < \infty$, then the dual space of H^p is H^q for $\dfrac{1}{p} + \dfrac{1}{q} = 1$, $1 < p, q < \infty$, that is, $(H^p)^* \cong H^q$. Therefore $B(H^p, \mathbb{R}) \cong (H^p)^* \cong H^q$. For $p = 1$, the dual of H^1 is the space BMO of functions of bounded means oscillations, periodic of period 2π for which

$$\|f\|_{BMO} = \sup_I \frac{1}{|I|} \int_I |f(t) - f_I| dt < \infty,$$

where $f_I = \dfrac{1}{|I|} \displaystyle\int_I f(t) dt$.

Comments: The dual of H^1 was an open problem for long time. It has been solved in the earlier 1970 by **Charles Fefferman**, which in the process of finding the dual of H^1, he find a characterization for H^1 well-known now as the **atomic decomposition** of H^1. This characterization had a very far reach implication in Harmonic Analysis, for example with the atomic decomposition came **De Souza** with the special atomic decomposition and the dyadic special decomposition, and consequently came **Yves Meyers** with the recreation of the wavelets.

CHAPTER 5

HAHN BANACH THEOREM

In this section we will study one of the most important theorem in functional analysis: the Hahn-Banach Theorem. For this section all the vector spaces are considered to be over the real numbers.

Analytic form of the Hahn-Banach Theorem

Definition 5.1. *Let X be a vector space, not necessarily normed and let $p : X \to \mathbb{R}$ be a function satisfying the following properties.*

i) $p(x) \geq 0, \quad \forall x \in X$

ii) $p(x+y) \leq p(x) + p(y), \quad \forall x, y \in X$

iii) $p(\alpha x) = \alpha p(x), \quad \forall x \in X, \ \alpha \in \mathbb{R}, \ \alpha > 0.$

A function $p : X \to \mathbb{R}$ satisfying all these properties is called a "convex functional". If only $ii)$ and $iii)$ are satisfied, p is called a "sublinear functional".

Note:

1. Each norm in a normed vector space is a convex functional *i.e.* $p(x) = \|x\|$.
2. Let $\psi : X \to \mathbb{R}$ be a bounded linear functional. Then p defined by $p(x) = \|\psi\| \|x\|_X$ is a convex functional.

Definition 5.2. *Let M be a proper subset of a vector space X and $f : M \to W$ be a function where W is a vector space. Then f can be extended to the space X if there is a function $F : X \to W$ so that $F|_M = f$ i.e. $F(x) = f(x), \ \forall x \in M$. The function F is said to be the extension of f from M to X.*

Example 5.1. Let $M = (-\infty, 1) \cup (1, \infty)$ and $X = \mathbb{R}$ and define $f : M \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & \text{if } x \in (-\infty, 1); \\ 1, & \text{if } x \in (1, \infty). \end{cases}$$

Then $F : \mathbb{R} \to \mathbb{R}$ defined by

$$F(x) = \begin{cases} 0, & \text{if } \ x \in (-\infty, 1]; \\ 1, & \text{if } \ x \in (1, \infty). \end{cases}$$

extend f from M to $\mathbb{R}$.

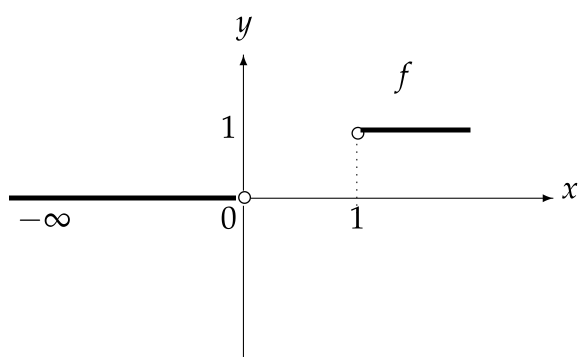
Example 5.2. Let $M = (-\infty, 0) \cup (1, \infty)$ and $X = \mathbb{R}$ and define $f : M \to \mathbb{R}$ by

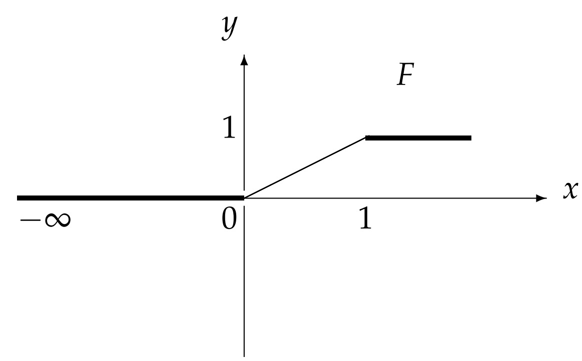
$$f(x) = \begin{cases} 0, & \text{if } x \in (-\infty, 0); \\ 1, & \text{if } x \in (1, \infty). \end{cases}$$

Then $F : \mathbb{R} \to \mathbb{R}$ defined by

$$F(x) = \begin{cases} 0, & \text{if } \ x \in (-\infty, 1]; \\ x, & \text{if } \ x \in [0, 1]; \\ 1, & \text{if } \ x \in [1, \infty). \end{cases}$$

extend f from M to $\mathbb{R}$.





Remark 5.1. *In general the extension of a function $f : M \to \mathbb{R}$ to a function $F : X \to \mathbb{R}$ ($M \subset X$) so that $F(x) = f(x)$, $\forall x \in M$, is not very interesting or important. However to extend f to F where F preserves the properties of f, is very important in Analysis.*

Example 5.3. Let $f(x) = \dfrac{\sin x}{x}$, $\ x \neq 0$, $M = \mathbb{R} \setminus \{0\}$. Define

$$F(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0; \\ 1, & x = 0. \end{cases}$$

$$G(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0; \\ 2, & x = 0. \end{cases}$$

Both F and G are extensions of f, but F is a continuous extension of f while G is simply an extension of f which is not continuous. Therefore F is an extension of f that preserve the continuity of f.

Remark 5.2. *The Hahn-Banach Theorem deal with an extension of a linear functional, defined in a linear subspace to the all space preserving some properties of the functional.*

Theorem 5.1. *(Hahn-Banach, The analytic form) Let M be a linear subspace of a vector space over the real numbers X and $p : X \to \mathbb{R}$ a sublinear functional, $\psi : X \to \mathbb{R}$ a linear functional so that $\psi(x) \leq p(x)$, $\forall x \in M$. Then,*

$i)$ There is a linear functional $F : X \to \mathbb{R}$ that is an extension of ψ i.e.
$F(x) = \psi(x)$, $\forall x \in M$.

$ii)$ $F(x) \leq p(x)$, $\forall x \in X$.

Below are some consequences of the Hahn-Banach Theorem.

Theorem 5.2. *Let ψ be a bounded linear functional defined on a linear subspace M of a normed vector space X, then,*

$i)$ There is a bounded linear functional $F : X \to \mathbb{R}$ that extend ψ to the all X i.e.
$F(x) = \psi(x)$, $\forall x \in M$.

$ii)$ $\|F\| = \|\psi\|$.

Comments about the proof of Theorem 5.2 If $M = \{0\}$ then set $\psi \equiv 0$ so that $F \equiv 0$ is the desired extension.
Now assume that $M \neq \{0\}$. So one can say there is an $x \in M$, $x \neq 0$. Moreover ψ is bounded, that is $|\psi(x)| \leq \|\psi\|\|x\|_X$, $\forall x \in M$. Define $p(x) = \|\psi\|\|x\|_X$, $\forall x \in X$. It is easy to see that p is a convex linear functional, therefore applying Theorem 5.1, we get Theorem 5.2.

Theorem 5.3. *Let $X \neq \{0\}$ be a normed vector space and $x_0 \in X$, then there is bounded linear functional ψ on X so that $\|\psi\| = 1$ and $\psi(x_0) = \|x_0\|_X$.*

Remark 5.3. *Theorem 5.3 is a very important one which claims that if $X \neq \{0\}$ then $X^* \neq \{0\}$.*

Comments about the proof of Theorem 5.3 Define $M = \{\alpha x_0;\ \alpha \in \mathbb{R}\}$, $x_0 \in X$ fixed, $x_0 \neq 0$. One can show that M is a linear subspace of X. Define

$$\begin{aligned} \psi : M &\longrightarrow \mathbb{R} \\ x &\longmapsto \psi(x) = \alpha\|x_0\|_X. \end{aligned}$$

ψ is a linear functional over M. Indeed,

$$\begin{aligned} \psi(x+y) = \psi(\alpha x_0 + \beta x_0) = \psi((\alpha+\beta)x_0) = (\alpha+\beta)\|x_0\|_X &= \alpha\|x_0\|_X + \beta\|x_0\|_X \\ &= \psi(x) + \psi(y). \end{aligned}$$

Moreover $\psi(\alpha x) = \alpha\psi(x)$, $\forall x \in M$, $\alpha \in \mathbb{R}$. Also note that

$$|\psi(x)| = |\alpha\|x_0\|_X| = \|\alpha x_0\|_X = \|x\|_X \tag{5.1}$$

Therefore $\|\psi\| \leq 1$. Indeed, $\|\psi\| = 1$, otherwise there is a constant $k < 1$ so that $|\psi(x)| \leq k\|x\|_X$, $\forall x \in M$, which contradict (5.1). On the other hand, $\psi(x_0) = \|x_0\|_X$ and applying Theorem 5.1, we get the conclusion of Theorem 5.3.

Remark 5.4. *If $x \in X$ so that $\forall \psi \in X^*$, $\psi(x) = 0$, then $x = 0$.*

Remark 5.5. *Let X be a normed vector space and*

$$\begin{aligned} F : X &\longrightarrow X^{**} \\ x &\longmapsto F_x \end{aligned}$$

where F_x is defined by $F_x(\psi) = \psi(x)$, $\forall \psi \in X^*$ and X^{**} is the dual of X^*, then, $\|F_x\|_{X^{**}} = \|x\|_X$ i.e. F is an isometry.

Proof. Note that $F_x : X^* \longrightarrow \mathbb{R}$. Then $|F_x(\psi)| = |\psi(x)| \leq \|\psi\|\|x\|_X$. Therefore,

$$\|F_x\| \leq \|x\|_X \tag{5.2}$$

On the other hand, if $x \neq 0$, there exists $\psi \in X^*$ such that $\|\psi\|_{X^*} = 1$ and $\psi(x) = \|x\|_X$, this follows from Theorem 5.3. But $|F_x(\psi)| = |\psi(x)| = \|x\|_X = \|x\|_X \|\psi\|_{X^*}$ and so

$$\|F_x\| \geq \|x\|_X \tag{5.3}$$

Putting (5.2) and (5.3) together we get $\|F\|_{X^{**}} = \|x\|_X$, so F is an isometry. $\square$

Remark 5.6. *Let c be the space of convergent sequences and l_∞ the space of bounded sequences, both with the supremun norm i.e. $(c, \|.\|_\infty)$ and $(l_\infty, \|.\|_\infty)$. We are going to show that not of all the bounded linear functional on l_∞ are of the form $\psi(x) = \sum_{n=1}^{\infty} x_n y_n$ with $y = (y_n) \in l_1$.*

Proof. We define $\varphi : c \longrightarrow \mathbb{R}$ as $\varphi(x) = \lim_{n\to\infty} x_n$. φ is a linear functional. Moreover

$$|\varphi(x)| = |\lim_{n\to\infty} x_n| \leq \sup_{n\geq 1} |x_n| = \|x\|_\infty \ \textit{and} \ \|\varphi\|_{c^*} \leq 1 \tag{5.4}$$

If we take $x = (x_n)$ an increasing sequence, then

$$|\varphi(x)| = |\lim_{n\to\infty} x_n| = \sup_{n\geq 1} |x_n| = \|x\|_\infty \ \textit{and so} \ \|\varphi\|_{c^*} \geq 1 \tag{5.5}$$

(5.4) and (5.5) together imply $\|\varphi\|_{c^*} = 1$. Note that $c \subset l_\infty$ and by Theorem 5.1(Hahn-Banach Theorem), φ can be extended to all l_∞, indeed there is $\phi : l_\infty \longrightarrow \mathbb{R}$ so that $\phi|_c = \varphi$ Now if $\phi(x) = \sum_{n=1}^{\infty} a_n x_n$ for some $a = (a_n) \in l_1$. Then for $x = e_i$, $i = 1, 2, 3, ...$ where $e_1 = (1, 0, ..., 0, ...)$, $e_2 = (0, 1, 0, ..., 0, ...)$,... or $e = (1, 1, ..., 1, 1, ...)$, we have

$$\phi(e) = \varphi(1) = \lim_{n\to\infty}(e) = 1$$

and

$$\phi(e_k) = \varphi(e_k) = \lim_{k\to\infty} e_k = 1.$$

Therefore

$$1 = \phi(e) = \sum_{n=1}^{\infty} a_n.$$

On the other hand

$$0 = \phi(e_k) = \sum_{k=1}^{\infty} a_k e_k = a_k, \ \ \forall k.$$

So we have

$$1 = \sum_{n=1}^{\infty} a_n = 0$$

which is a contradiction. $\square$

Geometric form of the Hahn-Banach Thorem

Definition 5.3. *A hyperplane H in a normed vector space X is defined as $H = \{x \in X : \ \psi(x) = c\}$ where ψ is a non-zero linear functional on X.*

Example 5.4. Let $X = \mathbb{R}$, define $H = \{x \in \mathbb{R} : \ 3x = -2\}$ where $\phi(x) = 3x$. Then $H = \{-2/3\}$ is a single point.

Example 5.5. Let $X = \mathbb{R}^2$, define $H = \{(x, y) \in \mathbb{R}^2 : \ 3x + 4y = 5\}$. Here $\phi(x, y) = 3x + 4y$ and H is the line of equation: $3x + 4y = 5$.

Example 5.6. Let $X = \mathbb{R}^3$, define $H = \{(x, y, z) \in \mathbb{R}^3 : \ 3x - 2y + 5z = 1\}$. Here $\phi(x, y) = 3x - 2y + 5z$ and H is the plane of equation: $3x - 2y + 5z = 1$.

Theorem 5.4. *The hyperplane $H = \{x \in X : \ \psi(x) = \alpha\}$ is closed if and only if ψ is a bounded linear functional on X.*

Remark 5.7. *A consequence of Theorem 5.4 is that the closed hyperplane are generated for those linear functionals on the dual space.*

Definition 5.4. *Let $A \subset X$ and $B \subset X$. We say that the hyperplane $H = \{x \in X : \ \psi(x) = \alpha\}$ separates A and B in the general sense if*

1) $\psi(x) \leq \alpha, \ \ \forall x \in A$

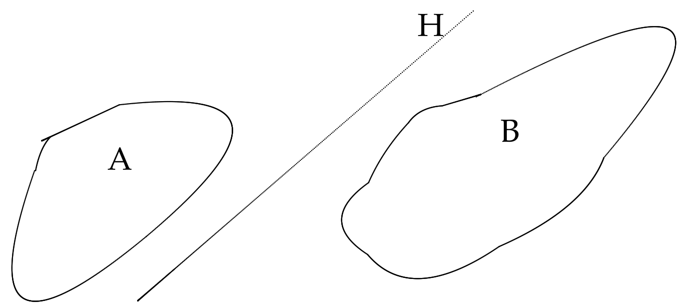
2) $\psi(x) \geq \alpha, \ \ \forall x \in B.$

Also we say that H separates A and B in the strict sense, if there is an $\epsilon > 0$ so that

1) $\psi(x) \leq \alpha - \epsilon, \ \ \forall x \in A$

2) $\psi(x) \geq \alpha + \epsilon, \ \ \forall x \in B.$

Remark 5.8. *Geometrically we say that A and B are separated by the hyperplane H, means A and B are situated in opposite side of H.*


H
A
B

CHAPTER 6

CLOSED GRAPH THEOREM

Definition 6.1. *Let X and Y be metric spaces and $T : X \longrightarrow Y$ a function, then we define the graph of T as*

$$G_T = \{(x, Tx) : \ x \in X\}.$$

Note that the graph of T is a subset of $X \times Y$, that is $G_T \subset X \times Y$ and $(x, y) \in G_T$ if and only if $y = Tx$.

Example 6.1. Let $X = [0, 1]$, $Y = \mathbb{R}$ and $T : [0, 1] \longrightarrow \mathbb{R}$ defined as $T(x) = x^2$. Then the graph of T is given by $G_T = \{(x, x^2) : \ x \in [0, 1]\}$.

Definition 6.2. *Let X, Y be normed vector spaces and $T : X \longrightarrow Y$ a function, then T is said to be a closed function if, the graph of T, G_T, is a closed subset of $X \times Y$.*

Remark 6.1. *The norm in the cartesian product space $X \times Y$ can be defined as one of the following:*

$$\|(x, y)\|_\infty = \max\{\|x\|_X, \|y\|_Y\},$$

$$\|(x, y)\|_1 = \|x\|_X + \|y\|_Y,$$

$$\|(x, y)\|_p = \left(\|x\|_p^p + \|y\|_p^p\right)^{1/p}.$$

These are equivalent norms in $X \times Y$.

Theorem 6.1. *Let X, Y be normed spaces and $T : X \longrightarrow Y$ any function, then T is closed if and only if $(x_n) \in D(T)$, domain of T, with $x_n \to x$ and $Tx_n \to Tx$, then we have*

i) $x \in D(T)$

ii) $Tx = y$

Remark 6.2. *We will give an example of a transformation that is i) linear, ii) closed and iii) not bounded.*

Example 6.2. Let $X = Y = C[0,1]$ with the supremum norm, that is,

$$\|f\|_\infty = \sup_{0 \le t \le 1} |f(t)|$$

$D = \{f \in C[0,1] : \ f' \ \textit{exists and is continuous in} \ [0,1]\}$ where f' is the derivative of f. Define $T : D \longrightarrow C[0,1]$ by $T(f) = f'$.

i) T is a linear transformation, this follows from the linearity of the derivative.

ii) T is closed. We will show that $G_T = \{(f, f') : \ f \in D\}$ is closed in $X \times Y$. Let $\|(f,g)\|_{X\times Y} = \|f\|_\infty + \|g\|_\infty$ be a norm in $X \times Y$ and $(f_n, Tf_n) \to (f, g)$ in $X \times Y$, that is

$$\|(f_n, Tf_n) - (f,g)\|_{X\times Y} = \|(f_n - f, Tf_n - g)\|_{X\times Y} = \|f_n - f\|_\infty + \|Tf_n - g\|_\infty \to 0,$$

as $n \to \infty$. Therefore $\|f_n - f\|_\infty \to 0$ and $\|Tf_n - g\|_\infty \to 0$ as $n \to \infty$ *i.e.* $\|f_n - f\|_\infty \to 0$, $\|f_n' - g\|_\infty \to 0$ as $n \to \infty$.

$$\|f_n' - g\|_\infty = \sup_{0\le t\le 1} |f_n'(t) - g(t)| \to 0 \ as \ n \to \infty$$

and so as the convergence is uniform, we have $\lim_{n\to\infty} f_n'(t) = g(t)$. So if we integrate

$$\begin{aligned} \int_0^t g(s)ds = \int_0^t \lim_{n\to\infty} f_n'(s)ds &= \lim_{n\to\infty} \int_0^t f_n'(s)ds \\ &= \lim_{n\to\infty} [f_n(t) - f(0)] \\ &= f(t) - f(0). \end{aligned}$$

which implies

$$f(t) = f(0) + \int_0^t g(s)ds,$$

therefore $f'(t) = g(t)$ that is, $Tf = g$ and so $(f,g) \in G_T$ which means that G_T is closed.

iii) T is not bounded. To see that consider $f_n(t) = t^n$. We have

$$\|f_n\|_\infty = \sup_{0\le t\le 1} |t^n| = 1,$$

$Tf_n(t) = f_n'(t) = nt^{n-1}$, so that

$$\|Tf_n\|_\infty = \sup_{0\le t\le 1} |nt^{n-1}| = n.$$

Thus T is unbounded.

Theorem 6.2. *(The Closed Graph Theorem) Let X, Y be Banach spaces, $T : X \to Y$ a linear transformation, then T is bounded if and only if G_T is closed.*

Remark 6.3. *If T is bounded, then it is immediate that $G_T = \{(x, Tx) : \ x \in X\}$ is closed in $X \times Y$. The other direction in Theorem 6.2 is the most important, which claims that if X, Y are Banach spaces and G_T is closed, then T is bounded.*

Remark 6.4. *A very nice consequence of the closed graph theorem is that given a normed space X endowed with two norms $\|.\|_X^1$ and $\|.\|_X^2$, if these norms make X a Banach space and are comparable, then these norms are equivalents, that is, if $(X, \|.\|_X^1)$ and $(X, \|.\|_X^2)$ are Banach spaces and $||x||_X^2 \le \kappa||x||_X^1$, there exist two positive constants α and β such that*

$$\alpha\|x\|_X^2 \le \|x\|_X^1 \le \beta\|x\|_X^2.$$

To see this, consider the identity mapping

$$i : (X, \|.\|_X^1) \longrightarrow (X, \|.\|_X^2)$$

but

$$\|x\|_X^2 \le \kappa\|x\|_X^1,$$

so

$$\|i(x)\|_X^2 \le \kappa\|x\|_X^1,$$

that is i is bounded; moreover

$$i^{-1} : (X, \|.\|_X^2) \longrightarrow (X, \|.\|_X^1)$$

is bounded by the closed graph theorem i.e.

$$\|i^{-1}(x)\|_X^1 \leq \beta \|x\|_X^2,$$

that is

$$\|x\|_X^1 \leq \beta \|x\|_X^2, \quad \textit{with} \quad \frac{1}{\kappa} = \alpha$$

since $i^{-1}(x) = x$ *and so*

$$\alpha \|x\|_X^2 \leq \|x\|_X^1 \leq \beta \|x\|_X^2.$$

Hence $\|.\|_X^1$ *and* $\|.\|_X^2$ *are equivalents.*

Example 6.3. Let $X = C[0,1]$ endowed with the norms

$$\|f\|_\infty = \sup_{0 \leq t \leq 1} |f(t)| \;\; \textit{and} \;\; \|f\|_1 = \int_0^1 |f(t)| dt.$$

Note that $\|f\|_1 \leq \|f\|_\infty$, but these norms are not equivalents; the reason is that $(C[0,1], \|.\|_1)$ is not a Banach space, see Example 2.7.

CHAPTER 7

OPEN MAPPING THEOREM

We now consider one of the basic and important result in functional analysis, which is "The open mapping theorem". This result allows us to conclude that under certain conditions, the inverse of an injective bounded transformation is bounded. We cannot conclude this in general, as we can see in the following example.

Example 7.1. Let X the set of all continuous functions $x(t)$ defined in the interval $[0,1]$ so that $x(0) = 0$. Define $M = \{x \in X : \ x'(t) \textit{ exists and is continuous}\}$. Endow X and M with the supremum norm *i.e.*

$$\|x\|_\infty = \sup_{0 \leq t \leq 1} |x(t)|.$$

Define the transformation $A : X \longrightarrow M$ by $(Ax)(t) = \displaystyle\int_0^t x(s)ds$, $0 \leq t \leq 1$
We can show that A is a bounded linear injective transformation. Indeed,

$$\|Ax\|_\infty = |\int_0^t x(s)ds| \leq \int_0^t |x(s)|ds \leq \|x\|_\infty.$$

Let $x_1(t)$ and $x_2(t)$ in X so that $Ax_1(t) = Ax_2(t)$, then $\displaystyle\int_0^t x_1(s)ds = \int_0^t x_2(s)ds$ and so differentiating both sides with respect to t, we get $x_1(t) = x_2(t)$. Thus A is injective.
But $A^{-1} : M \longrightarrow X$ is not continuous, in fact if we take $x_n(t) = \sin nt$, then
$Ax_n(t) = \dfrac{1 - \cos nt}{n}$. Therefore $Ax_n(t) \to 0$ as $n \to \infty$ in M but $x_n(t)$ does not have the limit in X.

Remark 7.1. *The hypothesis for which the inverse is bounded, is the completeness of X and M with the supremum norm. But $(M, \|.\|_\infty)$ is not a complete space, since a sequence of polynomials can converge uniformly to a continuous function without the derivative being continuous. Indeed it can converge to a continuous function nowhere differentiable.*

Definition 7.1. *Let X, Y be any metric spaces and $T : X \longrightarrow Y$ a transformation. We say* *that T is an open transformation or open mapping if, for any open set A in X, $T(A)$ is an open set in Y; that is T maps open sets in X into open sets in Y.*

Theorem 7.1. *(Open Mapping Theorem) Let T be a linear transformation from X onto Y i.e. $T : X \longrightarrow Y$ where X and Y are Banach spaces. Then T is an open transformation; that is, if $D \subset X$ is an open set in X, then $T(D)$ is an open set in Y.*

Remark 7.2. *The open mapping theorem is the fundamental tool for proving the closed graph theorem 6.2.*

CHAPTER 8

UNIFORM BOUNDEDNESS PRINCIPLE

The theorem of the uniform bounded principle together with the closed graph, the open mapping and the Hahn-Banach theorems are the basic theorems in Analysis. This theorem basically claims that if we have a family of bounded linear transformations $\{T_\alpha\}_{\alpha\in I}$ from X to Y, where X and Y are Banach spaces *i.e.* $T_\alpha : X \longrightarrow Y$, $\alpha \in I$ such that the T_α's are pointwise bounded, *i.e.*

$$\sup_{\alpha\in I} \|T_\alpha x\|_Y < \infty, \quad \forall x \in X.$$

Then the T_α's are uniformly bounded, that is,

$$\sup_{\alpha\in I} \|T_\alpha\| < \infty.$$

Remark 8.1. *The completeness of X is essential. Indeed, consider the space X defined by $X = \{x = (x_n \text{ sequence so that } x_k = 0) \text{ except } for\ number\ of\ k\}$. Let X be endowed with the l_p-norm, $1 \le p \le \infty$, that is,*

$$\|x\|_X = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}.$$

Take $Y = \mathbb{R}$ the real numbers. Define $T_n : X \longrightarrow \mathbb{R}$ by $T_n x = n x_n$, $n = 1, 2, 3, \ldots$.
Note that $T_n x = 0$ for n large enough. Therefore $T_n x$'s are pointwise bounded, but $\|T_n\| = n$ which shows that (T_n) is not uniformly bounded. In fact the problem stands on the non completeness of $(X, \|.\|_X)$

Remark 8.2. *The uniform Boundedness principle is also well known as the Banach-Steinhaus theorem.*

Theorem 8.1. *(The Uniform Boundedness Principle) Let X and Y be Banach spaces, T_α :*

$X \longrightarrow Y$, $\alpha \in I$ where I is an index set, a bounded linear transformation so that

$$\sup_{\alpha\in I} \|T_\alpha x\|_Y < \infty, \quad \forall x \in X,$$

then

$$\sup_{\alpha\in I} \|T_\alpha\| < \infty.$$

Example 8.1. Let $X = \{a = (a_1, a_2, ..., a_n, 0, 0, ...)\}$. Define $T_n : X \longrightarrow l_2$ by

$$T_n(e_i) = \begin{cases} 0, & i \neq n; \\ ne_n, & i = n. \end{cases}$$

where $e_1 = (1, 0, 0, ...)$, $e_2 = (0, 1, 0, ...)$,...,$e_n = (0, ..., 0, 1, 0, ...)$,....Then $x \in X$ implies $x = \sum_{i=1} ka_i e_i$. Therefore $T_n x = 0$ if $n > k$, hence

$$\sup_{n\geq 1} \|T_n x\|_{l_2} < \infty,$$

but $\|T_n\| = n$ and so

$$\sup_{n\geq 1} \|T_n\| = \infty$$

Example 8.2. Theorem 8.1 is a tool to prove that there is a periodic continuous function f of period 2π in $[-\pi, \pi]$ so that the Fourier series diverges in some point.

We end this section with an important theorem in Real Analysis, namely the Radon-Nikodym Theorem.

Definition 8.1 (Absolutely Continuous Measures)**.**
Suppose ν is a measure and μ is a positive measure defined on a measurable space $(X, \mathcal{M})$. ν is said to be absolutely continuous with respect to μ and denoted by

$$\nu \ll \mu$$

if $\nu(E) = 0$ for every $E \in \mathcal{M}$ for which $\mu(E) = 0$.

Example 8.3. Suppose f is a μ-integrable function defined on a measure space

$(X, \mu, \mathcal{M})$. Then the set function defined as

$$\nu(E) = \int_E f d\mu \quad \text{for every } E \in \mathcal{M}$$

is absolutely continuous with respect to μ.

Definition 8.2 (σ Finite Measures). *A measure μ is said to be σ-finite on a measure space $(X, \mathcal{M})$ is there is countable family $\{X_n\}_{n\in\mathbb{N}}$ such that*

$$X = \cup_{n\in\mathbb{N}} X_n, \quad \text{and } \mu(X_n) < \infty.$$

Theorem 8.2 (Radon-Nikodym). *Let $(X, \mathcal{M})$ be a measurable space. Let ν and μ be a σ-finite measures on $(X, \mathcal{M})$ such that $\nu \ll \mu$. Then there is a measurable function $f : X \to [0, \infty)$ such that*

$$\nu(E) = \int_E f d\mu \quad \text{for any } E \in \mathcal{M}.$$

Moreover, the function f satisfying the above equality is uniquely defined up to a null set, that is, if g is another such function, then $f = g$ a.e.

Remark 8.3. *The relation $\nu(E) = \int_E f d\mu$ is at times denoted by $d\nu = f\dot{\mu}$.*

CHAPTER 9

EQUIVALENCE OF SOME BANACH SPACES

In this lecture we will discuss the equivalence of Banach spaces resulting in analytic characterizations of some spaces of real-valued functions defined on the boundary of the complex unit disc that is identified with the interval $[0, 2\pi]$ or any interval of length 2π with some spaces of analytic functions defined in the disc. This characterization gives us a link between complex analysis and real analysis. We do not currently have any book that deals with this subject in a systematic way.

We begin with the definition of equivalence of Banach spaces.

Definition 9.1. *We say that two Banach spaces X and Y are equivalent if there is a linear transformation $T : X \to Y$ which is bijective and*

$$N\|x\|_X \leq \|Tx\|_Y \leq M\|x\|_X \,,$$

where N and M are absolute constants and $\|\cdot\|_X$ means the norm in X. Moreover, if $\|Tx\|_Y = \|x\|_X$, then X and Y are said to be isometric.

Note: We denote this equivalence as $X \cong Y$.

9.1 $\mathbb{R}^n$, Lipschitz, and Special Atom Spaces

Example 9.1. *Let $X = \mathbb{R}^{nm}$ be the space of all $(x_1, x_2, \ldots, x_{nm})$ with n, m positive integers and $x_i \in \mathbb{R}$, $i = 1, \ldots, nm$, where $\mathbb{R}$ denotes the set of real numbers. Let $Y = M_{nm}$ be the space of real $n \times m$-matrices. It is well known that these two spaces are equivalent as Banach spaces with their usual norms.*

Theorem 9.2. $\mathbb{R}^{nm} \cong M_{nm}$.

Example 9.3. *Let $X = Lip_\alpha$, $0 < \alpha \leq 1$, the Lipschitz space*

$$f \in Lip_\alpha \;\Leftrightarrow\; f \text{ is continuous and } |f(x+h) - f(x)| \leq Mh^\alpha$$

with

$$\|f\|_{Lip_\alpha} = \sup_{h>0,x} \frac{|f(x+h) - f(x)|}{h^\alpha}, 0 < \alpha \leq 1.$$

Let $Y = \Lambda_\alpha$, $0 < \alpha < 2$, *the space defined by*

$$f \in \Lambda_\alpha \;\Leftrightarrow\; f \text{ is continuous and } |f(x+h) + f(x-h) - 2f(x)| \leq Mh^\alpha$$

with

$$\|f\|_{\Lambda_\alpha} = \sup_{h>0,x} \frac{|f(x+h) + f(x-h) - 2f(x)|}{h^\alpha}, 0 < \alpha < 2.$$

Theorem 9.4. *$Lip_\alpha \cong \Lambda_\alpha$ for $0 < \alpha < 1$*

Note: If $\alpha > 1$, then $Lip_\alpha = \{0\}$ while $\Lambda_\alpha \neq \{0\}$ for $\alpha \in [1, 2)$.
Note: We are considering real valued functions in Lip_α and Λ_α as defined on the interval $[0, 2\pi]$.

Example 9.5. *Let $X = B^p$ for $1 \leq p < \infty$, where $f \in B^p$ if f is defined in $[0, 2\pi]$ and f can be represented as*

$$f(t) = \sum_{n=1}^{\infty} c_n b_n(t)\ , \text{ where } \sum_{n=1}^{\infty} |c_n| < \infty;,$$

with

$$b_n(t) = \frac{1}{|I_n|^{1/p}} \left[\chi_{R_n}(t) - \chi_{L_n}(t)\right]\ ,$$

and I_n, R_n, L_n are intervals in $[0, 2\pi]$ with $I_n = R_n \cup L_n$ and $R_n \cap L_n = \emptyset$. We endow B^p with the following norm

$$\|f\|_{B^p} = \inf \sum_{n=1}^{\infty} |c_n|$$

where the infimum is taken over all possible representations of f. We can show that $\|\cdot\|_{B^p}$ is indeed a norm and $(B^p, \|\cdot\|_{B^p})$ is a Banach space.

Let $Y = J^p$, $1 \leq p < \infty$ where $f \in J^p$ if f is defined in the interval $[0, 2\pi]$ and f can be represented as

$$f(t) = \sum_{n=1}^{\infty} \alpha_n \beta_n(t)\ , \text{ where } \sum_{n=1}^{\infty} |\alpha_n| < \infty\ ,$$

with

$$\beta_n(t) = \frac{1}{|I_n|^{1/p}} \chi_{I_n}(t)$$

and I_n are intervals in $[0, 2\pi]$. We endow J^p with the norm

$$\|f\|_{J^p} = \inf \sum_{n=1}^{\infty} |\alpha_n| < \infty$$

where the infimum is taken over all possible representations of f. We can show that $\|\cdot\|_{J^p}$ is indeed a norm and $(J^p, \|\cdot\|_{J^p})$ is a Banach space.

Theorem 9.6 (De Souza). *$B^p \cong J^p$ for $1 < p < \infty$.*

Note: The spaces B^p and J^p were introduced by De Souza in 1980 in his PhD dissertation.

9.2 L^p, Lorentz, and Special Atom Spaces Defined on General Measure

Definition 9.2. *Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space and suppose $p \geq 1$. A real-valued measurable function on X is said to be in $L^p(X, \mathcal{A}, \mu)$, or more briefly in L^p, if the function $|f|^p$ is integrable; that is, if*

$$\int_X |f(t)|^p \, d\mu(t) \; < \; \infty$$

The L^p space is endowed with a "norm"

$$\|f\|_p = \left(\int_X |f(t)|^p \, d\mu(t) \right)^{1/p} .$$

We wrote "norm" since $\|f\| = 0$ does not imply that f is the zero function but that $f = 0$ μ-almost everywhere; that is, $\mu\{x \in X : f(x) \neq 0\} = 0$. The difficulty vanishes if we agree to regard two members of L^p as the same if they are equal almost everywhere. This tells us that L^p is the quotient space with the equivalence relation $f(x) = 0$ a. e. With this in mind, $\|\cdot\|_p$ is a genuine norm.
Note: L^p is well-known as the Lebesgue space.

Definition 9.3. *Let f be a real-valued function defined on X. The decreasing rearrangement of f is the function f^* defined on $[0, \infty)$ by*

$$f^*(t) = \inf\{y > 0 : m(f, y) \leq t\} ,$$

where $m(f,y) = \mu\{x \in X : |f(x)| > y\}$ *is the* distribution *of the function* f.

The following are some properties of the decreasing rearrangement function:

(a) If $|f(t)| \leq |g(t)|$, then $f^*(t) \leq g^*(t)$.

(b) $(f \cdot g)^*(t) \leq f^*(t) \cdot g^*(t)$.

(c) $m(f,y) = m(f^*,y)$.

(d) $(kf)^*(t) = |k| f^*(t)$.

(e) $(f+g)^*(t) \leq f^*(at) + g^*(bt)$, $a+b=1,\ a,b>0$.

(f) $\lim_{t\to 0} f^*(t) = \|f\|_\infty$.

(g) $\|f\|_p = \|f^*\|_p$

Definition 9.4. *Given a measurable function* f *on a measure space* (X,μ) *and* $0 < p,q \leq \infty$, *define*

$$\|f\|_{pq} = \begin{cases} \left(\dfrac{q}{p} \displaystyle\int_0^\infty \left(t^{\frac{1}{p}} f^*(t) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} & \text{if } q < \infty, \\ \sup\limits_{t>0} t^{\frac{1}{p}} f^*(t) & \text{if } q = \infty. \end{cases}$$

The set of all measurable, real-valued functions f *defined on* X *with* $\|f\|_{pq} < \infty$ *is called* the Lorentz Space *with indices* p *and* q *and denoted by* $L(p,q)(X,\mu)$.

Notes:

1. Lorentz spaces were introduced by G. G. Lorentz in 1950 and 1951.

2. For some p,q, the quantity $\|f\|_{pq}$ is a norm and for others it is equivalent to a norm.

3. If $q=\infty$, then the space $L(p,\infty)$ is called the weak L^p space and it can easily be shown that L^p is continuously contained in $L(p,\infty)$; that is, $L^p \subset L(p,\infty)$ and $\|f\|_{p\infty} \leq M\|f\|_p$, $p \geq 1$, where M is an absolute constant. $\|f\|_{p\infty}$ is a norm.

4. If $p=q$, then $\|f\|_{pp} = \left(\displaystyle\int_0^\infty (f^*(t))^p \, d\mu(t) \right)^{1/p} = \|f\|_p$.

Note: In view of (4) in the preceding notes, the Lorentz spaces are generalization of the L^p spaces.
Note: Among the $L(p,q)$ Lorentz spaces, we are very much interested in $L(p,1)$ spaces which seem to have rich properties. For example, $L(p,1)$ is continuously contained in L^p; that is $L(p,1) \subseteq L^p$ and $\|f\|_p \leq C\|f\|_{p1}$, where C is an absolute constant.

Definition 9.5. *For $0 < \alpha \leq 1$ and μ a measure of sets in the interval $[0,2\pi]$, we define the space $B(\mu,\alpha)$ as*

$$B(\mu,\alpha) = \left\{ f : [0,2\pi] \to \mathbb{R} : \ f(t) = \sum_{n=1}^{\infty} c_n d_n(t), \ \sum_{n=1}^{\infty} |c_n| < \infty \right\} ,$$

where $d_n(t) = \dfrac{1}{\mu^{\alpha}(A_n)} \chi_{A_n}(t)$, A_n are μ-measurable sets in $[0,2\pi]$, c_n are real numbers, and χ_A is the characteristic function of the set A. We endow $B(\mu,\alpha)$ with the norm

$$\|f\|_{B(\mu,\alpha)} = \inf \sum_{n=1}^{\infty} c_n ,$$

where the infimum is taken over all possible representations of f.

Note: This space was introduced by De Souza.
Note: One can show that indeed $\|\cdot\|_{B(\mu,\alpha)}$ is a norm. Moreover, $(B(\mu,\alpha), \|\cdot\|_{B(\mu,\alpha)})$ is a Banach space.

Theorem 9.7 (De Souza). *$L(p,1) \cong B(\mu, 1/p)$ for $p > 1$.*

Definition 9.6. *Let $0 < \alpha \leq 1$ and let μ be a measure on sets in the interval $[0,2\pi]$. We define the space $A(\mu,\alpha)$ as*

$$A(\mu,\alpha) = \left\{ f : [0,2\pi] \to \mathbb{R} : \ f(t) = \sum_{n=1}^{\infty} c_n b_n(t), \ \sum_{n=1}^{\infty} |c_n| < \infty \right\} ,$$

where $b_n(t) = \dfrac{1}{\mu^{\alpha}(X_n)} [\chi_{A_n}(t) - \chi_{B_n}(t)]$, $X_n = A_n \cup B_n$, $A_n \cap B_n = \emptyset$, $\mu(A_n) = \mu(B_n)$, A_n, B_n are μ-measurable sets in $[0,2\pi]$, c_n are real numbers, and χ_E is the characteristic *function of the set E. We endow $A(\mu,\alpha)$ with the norm*

$$\|f\|_{A(\mu,\alpha)} = \inf \sum_{n=1}^{\infty} c_n \ ,$$

where the infimum is taken over all possible representations of f.

Note: The space $A(\mu,\alpha)$ was introduced by De Souza.
Note: One can show that $\|\cdot\|_{A(\mu,\alpha)}$ is a norm. Moreover, $(A(\mu,\alpha), \|\cdot\|_{A(\mu,\alpha)})$ is a Banach space.

Theorem 9.8 (De Souza)**.** *$A(\mu,\alpha) \cong B(\mu,\alpha)$ for $0 < \alpha < 1$.*

9.3 Generalized Lipschitz Spaces

Definition 9.7. *For $0 < \alpha < 1$ and μ a measure on sets of $[0,2\pi]$, we define the space $Lip(\mu,\alpha)$ as*

$$Lip(\mu,\alpha) = \left\{ f : [0,2\pi] \to \mathbb{R}, \frac{1}{\mu^{\alpha}(A)} \left| \int_A f(x) d\mu(x) \right| < M \right\}$$

where A is a μ-measurable set in $[0,2\pi]$. A norm is defined on $Lip(\mu,\alpha)$ as

$$\|f\|_{Lip(\mu,\alpha)} = \sup_{A} \frac{1}{\mu^{\alpha}(A)} \left| \int_A f(x) d\mu(x) \right| \ .$$

Note: This space was originally introduced by G. G. Lorentz in 1950.

Definition 9.8. *For $0 < \alpha \leq 1$ and μ a measure on sets of $[0,2\pi]$, we define the space $\Lambda(\mu,\alpha)$ as*

$$\Lambda(\mu,\alpha) = \left\{ f : [0,2\pi] \to \mathbb{R}, \frac{1}{\mu^{\alpha}(X)} \left| \int_A f(x) d\mu(x) - \int_B f(x) d\mu(x) \right| < M \right\}$$

for all μ-measurable sets X, A, B in $[0,2\pi]$ such that $X = A \cup B$, $A \cap B = \emptyset$. We endow $\Lambda(\mu,\alpha)$ with the norm

$$\|f\|_{\Lambda(\mu,\alpha)} = \sup_{\substack{X=A\cup B \\ A\cap B=\emptyset}} \frac{1}{\mu^{\alpha}(X)} \left| \int_A f(x) d\mu(x) - \int_B f(x) d\mu(x) \right| \ .$$

Note: The space $\Lambda(\mu, \alpha)$ was originally introduced by De Souza.
Fact: We can show that $\|\cdot\|_{Lip(\mu,\alpha)}$ and $\|\cdot\|_{\Lambda(\mu,\alpha)}$ are norms and $Lip(\mu, \alpha)$ and $\Lambda(\mu, \alpha)$ endowed with these norms are Banach spaces.
Note: $Lip(\mu, \alpha)$ and $\Lambda(\mu, \alpha)$ are natural generalizations of the Lipschitz spaces. In fact, if we take μ as the Lebesgue measure, $X = [x-h, x+h]$, $A = [x-h, x]$, $B = (x, x+h]$, and $\mu^\alpha(X) = (2h)^\alpha$. Then for f differentiable, we get

$$\frac{1}{\mu^\alpha(X)}\left|\int_A f'(x)d\mu(x) - \int_B f'(x)d\mu(x)\right| = \left|\frac{f(x+h)+f(x-h)-2f(x)}{(2h)^\alpha}\right| .$$

Theorem 9.9 (De Souza)**.** *$Lip(\mu, \alpha) \cong \Lambda(\mu, \alpha)$ for $0 < \alpha < 1$.*

9.4 Weighted Special Atom Spaces and Weighted Lipschitz Spaces

Definition 9.9. *We will define a weight function ρ as $\rho : [0, 2\pi] \to [0, 2\pi]$ which is nondecreasing and with $\rho(0) = 0$. Several additional conditions on the weight function will be needed:*

1. *ρ is Dini if*
$$\int_0^h \frac{\rho(t)}{t}dt \le c\rho(h)$$
for $h > 0$ and an absolute constant c.

2. *ρ is called almost decreasing if $\rho(t) \le c\rho(s)$ whenever $s \le t$, where c is an absolute constant.*

3. *ρ is in the class b_p for $p \ge 1$ if*
$$\int_h^{2\pi} \frac{\rho(t)}{t^{p+1}} \le c\frac{\rho(h)}{h^p}$$
where $h > 0$ and c is an absolute constant.

Definition 9.10. *We define the weighted special atom space B_ρ, where ρ is a weight function, as*

$$B_\rho = \left\{ f : [0, 2\pi] \to \mathbb{R}; f(t) = \sum_{n=1}^{\infty} c_n b_n(t); \sum_{n=1}^{\infty} |c_n| < \infty \right\} ,$$

where

$$b_n(t) = \frac{1}{\rho(|I_n|)} \left[\chi_{R_n}(t) - \chi_{L_n}(t)\right] \ ,$$

and I_n is an interval of length $|I_n|$ with right and left half intervals R_n and L_n such that $I_n = R_n \cup L_n$, $L_n \cap R_n = \emptyset$. B_ρ is endowed with the norm

$$\|f\|_{B_\rho} = \inf \sum_{n=1}^{\infty} |c_n| \ ,$$

where the infimum is taken over all possible representations of f.

Note: If we take $\rho(t) = t^{1/p}$ for $p > 1$, then $B_\rho = B^p$ defined in Example 9.5.

Definition 9.11. *We define the weighted space J_ρ, where ρ is a weight function, as*

$$B_\rho = \left\{ f : [0, 2\pi] \to \mathbb{R}; f(t) = \sum_{n=1}^{\infty} \frac{c_n}{\rho(|I_n|)} \chi_{I_n}(t); \sum_{n=1}^{\infty} |c_n| < \infty \right\} \ ,$$

where I_n is an interval in $[0, 2\pi]$ of length $|I_n|$. J_ρ is endowed with the norm

$$\|f\|_{J_\rho} = \inf \sum_{n=1}^{\infty} |c_n| \ ,$$

where the infimum is taken over all possible representations of f.

Note: If we take $\rho(t) = t^{1/p}$ for $p > 1$, then $J_\rho = J^p$ defined in Example 9.5.

Theorem 9.10 (Bloom and De Souza)**.** *If $\rho \in b_1$, then $B_\rho \cong J_\rho$.*

Definition 9.12. *We define the weighted Lipschitz space as the set*

$$Lip_\rho = \{ f : [0, 2\pi] \to \mathbb{R};\ continuous\ ; |f(x+h) - f(x)| \le C\rho(|h|) \} \ .$$

We endow Lip_ρ with the norm

$$\|f\|_{Lip_\rho} = \sup_{h>0,\ x} \left| \frac{f(x+h) - f(x)}{\rho(h)} \right| \ .$$

Definition 9.13. *We define the second difference weighted Lipschitz spaces as*

$$\Lambda_\rho = \{ f : [0, 2\pi] \to \mathbb{R};\ continuous\ ; |f(x+h) + f(x-h) - 2f(x)| \le c\rho(|h|) \} \ .$$

We endow Λ_ρ *with the norm*

$$\|f\|_{\Lambda_\rho} = \sup_{h>0,\ x} \left| \frac{f(x+h)+f(x-h)-2f(x)}{\rho(2h)} \right| .$$

Note:

1. If $\rho(t) = t^\alpha$ for $0 < \alpha < 1$, then $Lip_\rho = Lip_\alpha$.
2. If $\rho(t) = t^\alpha$ for $0 < \alpha \leq 1$, then $\Lambda_\rho = \Lambda_\alpha$.

Theorem 9.11 (Bloom and De Souza)**.** *If* ρ *is Dini and in the class* b_1*, then* $Lip_\rho \cong \Lambda_\rho$.

9.5 Analytic Characterization of some Banach Spaces

In this section, we will see examples of equivalence between spaces of analytic functions defined on the unit disc with spaces of real-valued functions defined on the perimeter of the unit disc which is identified by the interval $[0, 2\pi]$ from the Banach space point of view. To observe this, define X and Y as

$$X = \{F : \mathbb{D} \to \mathbb{C};\ \text{analytic in } \mathbb{D} \text{ satisfying some property } \mathscr{P}\}$$

and

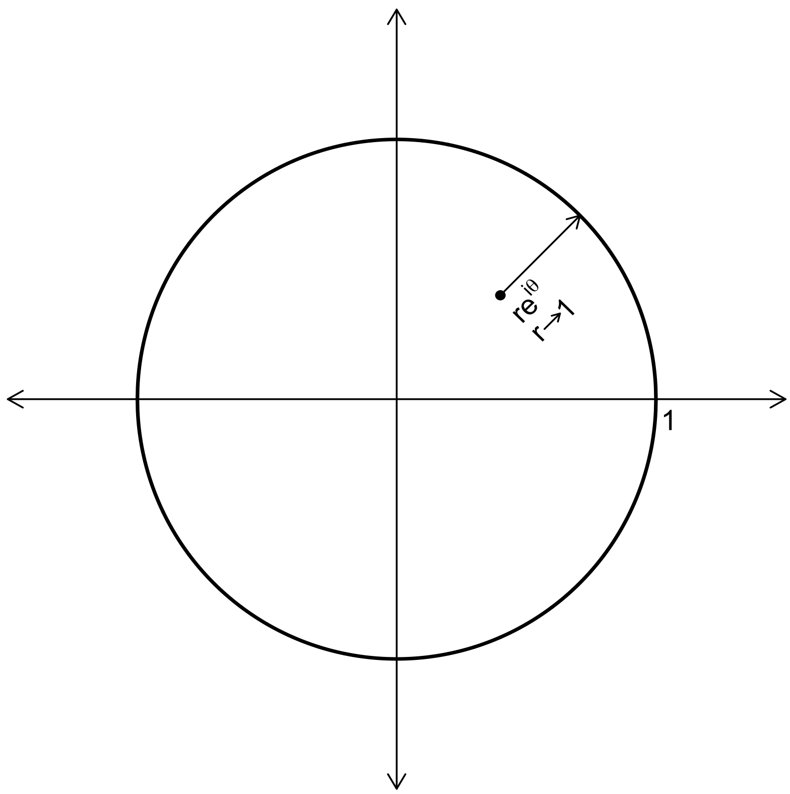
$$Y = \{f : [0, 2\pi] \to \mathbb{R};\ \text{periodic satisfying some property } \mathscr{Q}\}$$

where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{C}$ is the set of complex numbers.

Question: What are the properties $\mathscr{P}$ and $\mathscr{Q}$ such that X and Y are equivalent as Banach spaces?

Geometrically:
We consider $F \in X$ and $f \in Y$ in the following manner:

If $F \in X$, then define $f(\theta) = \lim_{r\to 1} Re\ F(re^{i\theta})$ a.e.



Question: What property does X need to possess in order to guarantee the existence of this limit almost everywhere?

For $f \in Y$, define

$$F(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} f(t)dt \ , \quad z = re^{i\theta} \ ,$$

where

$$\frac{e^{it}+z}{e^{it}-z}=\frac{1-r^2}{1-2\cos(\theta-t)+r^2}+i\frac{2r\sin(\theta-t)}{1-2r\cos(\theta-t)+r^2}=P(r,\theta-t)+iQ(r,\theta-t)$$

P is the Poisson kernel and Q is the conjugate Poisson kernel. Consequently,

$$\lim_{r\to 1}F(re^{i\theta})=\lim_{r\to 1}\left[Re\ F(re^{i\theta})+i\ Im\ F(re^{i\theta})\right]=f(\theta)+\tilde{f}(\theta)\ ,$$

where $\tilde{f}(\theta)$ is the conjugate function of f. Moreover,

$$\lim_{r\to 1}Re(iF(re^{i\theta}))=-\tilde{f}(\theta)\ .$$

Therefore,$F\in X$ and $f\in Y$ defined as above must have

$$f\in Y\ \Rightarrow\ \tilde{f}\in Y\ ;$$

that is, Y must be invariant under the conjugate function.
Recall that $\tilde{f}$ can be represented by

$$\tilde{f}(x)=pv\ \frac{1}{\pi}\int_0^{2\pi}\frac{f(t)}{2\tan\left(\frac{t-x}{2}\right)}dt\ .$$

As $\tan t\approx t$ as $t\to 0$, we have

$$\tilde{f}(x)=pv\ \frac{1}{\pi}\int_0^{2\pi}\frac{f(t)}{t-x}dt\ .$$

$\tilde{f}$ is known as the *conjugate operator* or the *Hilbert transform*.

Note: pv in the definition of $\tilde{f}$ represents the Cauchy principal value. That is,

$$\tilde{f}(x)=pv\ \frac{1}{\pi}\int_0^{2\pi}\frac{f(t)}{t-x}dt=\lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{\epsilon\le|t|\le\pi}\frac{f(t-x)}{t}dt$$

Note: From the above, we see that for a real-valued function space defined on the boundary of the disc to be equivalent to an analytic function space on the disc, it must be invariant under the Hilbert transform.

In fact, it is well known that there are functions $f \in L^1[0,2\pi]$ so that $\tilde{f} \notin L^1[0,2\pi]$. It is also known that if $f \in L^1[0,2\pi]$, then $\tilde{f}$ exists almost everywhere and

$$\alpha\mu\{x \in [0,2\pi] : |\tilde{f}(x) > \alpha\} \leq c\|f\|_1$$

where c is a constant independent of f and μ is the Lebesgue measure. Again, in order for X to be equivalent to Y, Y must be invariant under the Hilbert transform.

Final Comment: The type of question about the equivalence of X and Y leads to a difficult question:

What spaces on $[0,2\pi]$ are invariant under the Hilbert transform?

Example 9.12. *Take $X = H^p(\mathbb{D})$, $1 \leq p < \infty$, the Hardy space. $F \in H^p(\mathbb{D}) \Leftrightarrow F$ is analytic and*

$$\|F\|_{H^p} = \lim_{r\to 1}\left(\int_0^{2\pi} |F(re^{i\theta})|^p d\theta\right)^{1/p} < \infty\,.$$

Take $Y = L^p(\partial\mathbb{D})$, where $\partial\mathbb{D}$ is the boundary of the unit disc $\mathbb{D}$.

$$f \in L^p(\partial\mathbb{D}) \;\Leftrightarrow\; f : \partial\mathbb{D} \to \mathbb{R} \text{ and } \|f\|_p = \left(\int_{\partial\mathbb{D}} |f(t)|^p dt\right)^{1/p} < \infty$$

for $p > 1$.

Theorem 9.13. *$H^p(\mathbb{D}) \cong L^p(\partial\mathbb{D})$ for $1 < p < \infty$.*

Note: *The equivalence in Theorem 9.13 is a direct result of a famous theorem due to Marcel Riesz:*

Theorem 9.14 (M. Riesz). *If $f \in L^p$, then $\tilde{f} \in L^p$. Moreover, $\|\tilde{f}\|_p \leq M\|f\|_p$ for $1 < p < \infty$, where M is an absolute constant.*

Note: *This theorem does not hold for $p = 1$ or $p = \infty$. $H^p(\mathbb{D})$ is an analytic characterization of the space L^p for $p > 1$.*

Example 9.15. *Let*

$$X = S_\alpha = \{F : \mathbb{D} \to \mathbb{C}, \text{ analytic }, \|F\|_X = \sup_{|z|<1} (1-|z|)^{1-\alpha}|F'(z)| < \infty\}$$

for $0 < \alpha < 1$. *Let*

$$Y = \{f : \partial\mathbb{D} \to \mathbb{R}, \|f\|_Y = \sup_{h>0,x} \frac{|f(x+h) - f(x)|}{h^\alpha} < \infty\}$$

for $0 < \alpha < 1$. *Note that* $Y = Lip_\alpha$. *Here* F' *denotes the first derivative of* F.

Theorem 9.16 (Hardy-Littlewood). $X \cong Lip_\alpha$, $0 < \alpha < 1$.
$F \in X \;\Leftrightarrow\; f \in Y$. *Moreover,* $\|F\|_X \cong \|f\|_Y$, *where*

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} f(t) dt$$

.

Note: *It is well known that* $\|\cdot\|_X$ *is a norm and* $(X, \|\cdot\|_X)$ *is a Banach space. The space* X *is the analytic characterization of the Lipschitz space* Lip_α *for* $0 < \alpha < 1$.

Example 9.17. *Let*

$$X = Z_1 = \{F : \mathbb{D} \to \mathbb{C}, \text{ analytic }, \|F\|_X = \sup_{|z|<1} (1-|z|)^2 |F''(z)| < \infty\} .$$

Let

$$Y = \{f : \partial\mathbb{D} \to \mathbb{R}, \|f\|_Y = \sup_{h>0,x} \frac{|f(x+h) + f(x-h) - 2f(x)|}{2h} < \infty\}$$

Here F'' *denotes the second derivative of* F.

Theorem 9.18 (Antony Zygmund). $X \cong Y$.

Note: *The space* Y *is usually denoted by* Λ_* *and is known as the Zygmund class. It is very important in the study of the theory of Fourier series and approximation theory.*
For $0 < \alpha < 1$, $\Lambda_* \subseteq Lip_\alpha$ *and if* $\alpha < \beta$ *we have*

$$\Lambda_* \subseteq Lip_\beta \subseteq Lip_\alpha .$$

The space Λ_* is considered the limit of Lip_α as $\alpha \to 1$.
X is an analytic characterization of the Zygmund class Λ_*.

Example 9.19. *Take $X = H^1(\mathbb{D})$, the Hardy space defined in Example 9.12 for $p = 1$. Take*

$$Y = A = \{f : [0, 2\pi] \to \mathbb{R}, f(t) = \sum_{n=1}^{\infty} c_n a_n(t); \sum_{n=1}^{\infty} |c_n| < \infty\},$$

where c_n's are real numbers and a_n's are functions $a_n : [0, 2\pi] \to \mathbb{R}$ so that

1. *supp $a_n \subseteq I_n$, where $I_n \subseteq [0, 2\pi]$ is an interval,*
2. $|a_n(t)| \leq \dfrac{1}{|I_n|}$, *and*
3. $\displaystyle\int_{I_n} a_n(t)dt = 0.$

Endow A with the norm

$$\|f\|_B = \inf \sum_{n=1}^{\infty} |c_n|$$

where the infimum is taken over all possible representations of f.
We can show that $\|\cdot\|_A$ is a norm and $(A, \|\cdot\|_A)$ is a Banach space.

Theorem 9.20 (C. Fefferman and R. Coifman)**.** *$X \cong A$.*

Note: *A long standing open problem was finding the set of all bounded linear functional on $H^1(\mathbb{D})$. That is, to find the dual space of $H^1(\mathbb{D})$ denoted by $(H^1(\mathbb{D}))^*$. In 1972, Charles Fefferman showed that the dual space of $H^1(\mathbb{D})$ was the space BMO, the space of function of bounded mean oscillation. BMO is defined by*

$$g \in BMO[0, 2\pi] \iff \|g\|_{BMO} = \sup_I \frac{1}{|I|} \int_I |g(t) - g_I| dt < \infty,$$

where $g_I = \dfrac{1}{|I|} \displaystyle\int_I g(t)dt$.
The space A is well known and denoted by $A = Re\ H^1$, the real characterization of the Hardy space $H^1(\mathbb{D})$.

Example 9.21. *Take*

$$X = \{F : \mathbb{D} \to \mathbb{C}, \text{ analytic}, \|F\|_X = \int_0^1 \int_0^{2\pi} |F'(re^{i\theta})| d\theta dr < \infty\}.$$

Take

$$Y = B = \{f : [0, 2\pi] \to \mathbb{R}, f(t) = \sum_{n=1}^{\infty} c_n b_n(t); \sum_{n=1}^{\infty} |c_n| < \infty\} ,$$

where c_n are real numbers and

$$b_n(t) = \frac{1}{|I_n|} \left[\chi_{R_n}(t) - \chi_{L_n}(t)\right] ,$$

and $I_n \subseteq [0, 2\pi]$ is an interval of length $|I_n|$ with right and left half intervals R_n and L_n such that $I_n = R_n \cup L_n$, $L_n \cap R_n = \emptyset$.

Endow B with the norm

$$\|f\|_A = \inf \sum_{n=1}^{\infty} |c_n|$$

where the infimum is taken over all possible representations of f.
We can show that $\|\cdot\|_B$ is a norm and $(B, \|\cdot\|_B)$ is a Banach space.

Theorem 9.22 (G. De Souza and G. Sampson)**.** *$X \cong B$.*

Note: *De Souza introduced the space B in his PhD dissertation in 1980. He called these spaces "special atom spaces". Starting with Guido Weiss and Yves Meyer, these spaces are later referred to as De Souza's spaces.*

Another long standing open problem involved finding a real characterization of the space X. This was resolved by De Souza and Sampson in 1983 in a paper published in the Journal of the London Mathematical Society under the title "A real characterization of the pre-dual of Bloch functions".

The space X is called an analytic characterization of the space B and, on the other hand, B is called a real characterization of the space X.

Example 9.23. *Take*

$$X = I^p = \{F : \mathbb{D} \to \mathbb{C}, \text{ analytic }, \|F\|_X = \frac{1}{2\pi} \int_0^1 \int_0^{2\pi} |F'(re^{i\theta})|(1-r)^{1/p-1} d\theta dr < \infty\} .$$

Take $Y = B^p$, $1 < p < \infty$, the space introduced in Example 9.5. That is

$$Y = B^p = \{f : [0, 2\pi] \to \mathbb{R}, f(t) = \sum_{n=1}^{\infty} c_n b_n(t); \sum_{n=1}^{\infty} |c_n| < \infty\} ,$$

where c_n's are real numbers and

$$b_n(t) = \frac{1}{|I_n|^{1/p}} \left[\chi_{R_n}(t) - \chi_{L_n}(t)\right] ,$$

and $I_n \subseteq [0, 2\pi]$ is an interval of length $|I_n|$ with right and left half intervals R_n and L_n such that $I_n = R_n \cup L_n$, $L_n \cap R_n = \emptyset$.

Theorem 9.24 (G. De Souza)**.** $B^p \cong I^p$.

Note: *The space I^p is an analytic characterization of the space B^p. Also, in a paper that appeared in the Proceedings of the American Mathematical Society in 1985, we have shown that the Besov space $\Lambda(1-1/p, 1, 1)$ is equivalent to B^p; that is, $\Lambda(1-1/p, 1, 1) \cong B^p$. Recall that $\Lambda(1-1/p, 1, 1)$ is the set of functions defined on $[0, 2\pi]$ so that*

$$\|f\|_{\Lambda(1-1/p,1,1)} = \int_0^{2\pi}\int_0^{2\pi} \frac{|f(x)-f(y)|}{|x-y|^{2-1/p}} dxdy < \infty .$$

Theorem 9.25. *$I^p \cong \Lambda(1-1/p, 1, 1)$ for $1 < p < \infty$.*

9.6 Analytic Characterization of some Weighted Banach Spaces

In this section, we continue our study of characterization of Banach spaces. However, this time our interest is in weighted Banach spaces, specifically for the weighted special atom spaces and weighted Lipschitz spaces introduced in Section 9.4.

Example 9.26. *Let*

$$X = S_\rho = \{F : \mathbb{D} \to \mathbb{C},\ analytic\ , \|F\|_{S_\rho} = \sup_{|z|<1} \frac{1-|z|}{\rho(1-|z|)} |F'(z)| < \infty\} .$$

Take $Y = Lip_\rho$.
Here the weight function ρ is Dini and $\rho \in b_1$.

Theorem 9.27 (Bloom and De Souza)**.** $S_\rho \cong Lip_\rho$.

Example 9.28. *Take*

$$X = Z_\rho = \{F : \mathbb{D} \to \mathbb{C},\ analytic\ , \|F\|_{S_\rho} = \sup_{|z|<1} \frac{(1-|z|)^2}{\rho(1-|z|)} |F''(z)| < \infty\} ,$$

Take $Y = \Lambda_\rho$
The weight function ρ *is Dini,* $\rho \in b_2$*, and* $\dfrac{\rho(t)}{t^2}$ *is bounded below as* $t \to \infty$.

Theorem 9.29. $Z_\rho \cong \Lambda_\rho$.

Example 9.30. *Take*

$$X = I_\rho = \left\{ F : \mathbb{D} \to \mathbb{C},\ \textit{analytic}\ , \|F\|_X = \frac{1}{2\pi} \int_0^1 \int_0^{2\pi} |F'(re^{i\theta})| \frac{\rho(1-r)}{1-r} d\theta dr < \infty \right\} .$$

Take $Y = B_\rho$
The weight function ρ *is Dini and* $\rho \in b_2$.

Theorem 9.31 (Bloom and De Souza)**.** $I_\rho \cong B_\rho$.

Note: The spaces S_ρ, Z_ρ, and I_ρ are the weighted analytic characterizations of the respective weighted spaces Lip_ρ, Λ_ρ, and B_ρ.

9.7 Bounded Operators on some Banach Spaces

In this section, we give examples of important operators in harmonic analysis that are bounded in some Banach space. Showing boundedness is possible by using characterizations of the given spaces given earlier in these notes. We start by giving some useful definitions.

Definition 9.14. *A linear operator or linear transformation from a vector space* X *to a vector space* Y *over the same field* F *is a function* $T : X \to Y$ *satisfying the condition*

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$

for all $x, y \in X$ *and* $\alpha, \beta \in F$.

Note: Some important estimators in harmonic analysis are not linear but behave like one. These operators are called quasilinear operators and are defined as

$$|T(\alpha x + \beta y)(t)| \leq K(|\alpha||T(x)(t)| + |\beta||T(y)(t)|)$$

for all t and some $K > 0$. For example, Carleson's maximal operator (defined below) and the Hardy-Littlewood maximal operator defined as

$$Mf(x) = \sup_{x \in I} \frac{1}{|I|} \int_I |f(x)| dx$$

are quasilinear operators.

Definition 9.15. *A linear or quasilinear operator $T : X \to Y$, where X, Y are normed spaces, is said to be bounded if*

$$\|Tx\|_Y \leq M \|x\|_X$$

where M is an absolute constant and $\|\cdot\|_X$ and $\|\cdot\|_Y$ are the respective norms in X and Y.

9.7.1 Carleson's Maximal Operator

The Carleson's maximal operator is defined as follows.

Definition 9.16. *Let f be a periodic function of period 2π and let $S_n(f, x)$ be the nth partial sum of the Fourier transform of f. Then Carleson's maximal operator is defined as*

$$Tf(x) = \sup_{n \geq 0} |S_n(f, x)|$$

A problem that stayed open for a long time was the so-called "Lusin's conjecture" that says

If $f \in L^2[0, 2\pi]$, then the Fourier series of f converges to f almost everywhere.

An Argentinian mathematician, Alberto Calderón, transformed the conjecture into a problem of operators. Indeed, he conjectured that

$$Tf(x) = \sup_{n \geq 0} |S_n(f, x)|$$

is bounded in $L^2[0, 2\pi]$ into $L^2[0, 2\pi]$ if and only if $S_n(f, x) \to f$ almost everywhere. But it was the Swedish mathematician, Lennart Carleson, who in 1966 proved

Theorem 9.32 (Carleson's Theorem)**.** $Tf(x) = \sup_{n \geq 0} |S_n(f, x)|$ *is bounded in $L^2[0, 2\pi]$ into $L^2[0, 2\pi]$.*

The original proof of this theorem was much too complicated for most people to understand. To simplify the argument, De Souza showed in 1984 that Carleson's maximal operator T is bounded in $\Lambda(1-1/p,1,1)$ into $L(p,1)$. Indeed we have

Theorem 9.33 (De Souza)**.** *The Carleson maximal operator $T : \Lambda(1-1/p,1,1) \to L(p,1)$ is bounded. Moreover, $\|Tf\|_{L(p,1)} \leq M\|f\|_{\Lambda(1-1/p,1,1)}$ for $p > 1$ where M is an absolute constant.*

The proof of this theorem was made possible due to the characterization of $\Lambda(1-1/p,1,1)$ with the space B^p as given in Theorem 9.24 and Theorem 9.25. Also, De Souza showed that

Theorem 9.34. *The Carleson maximal operator $T : B_\rho \to L_\phi$ is bounded. Moreover, $\|Tf\|_{L_\phi} \leq M\|f\|_{B_\rho}$, where M is an absolute constant and L_ϕ is the weighted $L(p,1)$ space that is defined as*

$$f \in L_\phi \quad \textit{if and only if} \quad \int_0^{2\pi} f^*(t)\frac{\phi(t)}{t}dt < \infty$$

where ϕ is a weight function satisfying some conditions.

A consequence of Theorems 9.33 and 9.34 is the almost everywhere convergence of the Fourier series of f in those spaces. Another interesting result is

Theorem 9.35 (De Souza)**.** *The Carleson maximal operator $T : L(p,1) \to L(p,1)$ is bounded. Moreover, $\|Tf\|_{L(p,1)} \leq M\|f\|_{L(p,1)}$ for $p > 1$ where M is an absolute constant.*

Again, the proof of this theorem was possible because of the characterization of the space $L(p,1)$ as given in Theorem 9.7.

9.7.2 Multiplication Operator

In this section we will deal with the multiplication operator on $L(p,1)$ and $\Lambda(1-1/p,1,1)$ for $p > 1$. The results are extended to the space $L(p,q)$.

Definition 9.17. *For a given function g, we define the multiplication operator T_g as $T_g(f) = f \cdot g$ which is understood as the pointwise multiplication $(G \cdot f)(x) = g(x)f(x)$.*

The following result was given by Shen in his Auburn University masters project:

Theorem 9.36 (Multiplication operator on $\Lambda(1-1/p,1,1)$)**.** *The multiplication operator $T : \Lambda(1-1/p,1,1) \to \Lambda(1-1/p,1,1)$, $p>1$, is bounded if and only if g is bounded almost everywhere and $\forall h \in (0,\pi)$, $\forall a \in [-\pi,\pi]$*

$$\frac{1}{h^{1/p}}\int_0^h\int_a^{a+h}\frac{|g(x+t)-g(x)|}{t^{2-1/p}}dxdt \le A < \infty\,.$$

Once again, note that this result was possible due to the characterization of $\Lambda(1-1/p,1,1)$ as B^p, $1<p<\infty$, as given in Theorem 9.24 and Theorem 9.25.

More recently, Kwessi, De Souza, Alfonso and Abebe, showed

Theorem 9.37 (Multiplication operator on $L(p,1)$)**.** *The multiplication operator $T_g : L(p,1) \to L(p',1)$ for $p' \ge p > 1$ is bounded if and only $g \in L^\infty$. Moreover, $\|Tg\| = \|g\|_\infty$.*

This result was previously found in 2008 by Arora, Datt and Verma on Lorentz-Bochner spaces (*Osaka J. Math.*). The approach of Kwessi et al. depended on a new characterization of L^∞ which they called M_1^p. This space is defined as $f \in M_1^p$ if and only if

$$\|f\|_{M_1^p} = \sup_{x>0}\left(\frac{1}{x^{1/p}}\int_0^x f^*(t)t^{1/p-1}dt\right) < \infty\,.$$

In fact, Kwessi et al. defined the space M_r^p which is the set of real-valued functions f defined on $[0,2\pi]$ such that

$$\|f\|_{M_r^p} = \sup_{x>0}\left(\frac{r}{px^{1/p}}\int_0^x\left(f^*(t)t^{1/p}\right)^r\frac{dt}{t}\right)^{1/r} < \infty,$$

where $p,r>1$. They show that this space is a characterization of some weak L^p space, specifically $M_r^p \cong L(pr',\infty)$ where $1/r+1/r'=1$.

The following result on $L(p,q)$ can be obtained using Theorem 9.37 and the Marcinkiewicz interpolation theorem.

Theorem 9.38 (Multiplication Operator on $L(p,q)$)**.** *The multiplication operator $T_g : L(p,q) \to L(p,q)$ is bounded if and only if $g \in L^\infty$ for $1<p\le\infty$, $1<q\le\infty$. Moreover, $\|T_g\| = \|g\|_\infty$.*

Since $L^\infty \cong M_1^p \subseteq M_r^p$ for $r>1$, $T_g : L(p,q) \to L(p,q)$ is bounded implies that $g \in M_r^p$. A question of interest is

When does $g \in M_r^p$ imply that $T_g : L(p_1, q_1) \to L(p_2, q_2)$ is bounded?

Noting that M_r^p is some weak L^p space which is a special case of Lorentz spaces, the question can be posed in a more general manner as

What are r and s so that $f \in L(p_1, q_1)$ and $g \in L(p_2, q_2)$ imply that $T_g \in L(r, s)$?

Theorem 9.39 (Kwessi et al.). *If $f \in L(p_1, q_1)$ and $g \in L(p_2, q_2)$, where $1 < p_1, p_2, q_1, q_2 < \infty$, then $T_g \in L(r, s)$, where $1/r = 1/p_1 + 1/p_2$ and $1/s = 1/q_1 + 1/q_2$. Moreover, $\|T_g f\|_{L(r,s)} \leq \|f\|_{L(p_1,q_1)} \|g\|_{L(p_2,q_2)}$.*

However, we still have not addressed the case where $g \in M_r^p$ since the theorem does not apply for $q_2 = \infty$. A result that includes $q_2 = \infty$ is given by Kwessi et al. as follows.

Theorem 9.40. *If $g \in M_r^p$, then $T_g : L(q, s) \to L\left(\frac{pqr'}{pr'+q}, s\right)$ is bounded, where $\frac{1}{r} + \frac{1}{r'} = 1$ and for $s > 0$ and $p, q > 1$.*

9.7.3 Composition Operator

In this section, we will briefly discuss composition operators on Lorentz spaces $L(p, q)$.

Definition 9.18. *Let μ be a measure on $[0, 2\pi]$ and $g : [0, 2\pi] \to [0, 2\pi]$ be a μ-measurable function such that $\mu(g^{-1}(A)) \leq c\mu(A)$, where A is a μ-measurable set in $[0, 2\pi]$, c is an absolute constant, and $g^{-1}(A)$ is the pre-image of the set A. The composition operator C_g is defined as $C_g(f) = f \circ g$.*

Theorem 9.41. *The composition operator $C_g : L(p, q) \to L(p, q)$ is bounded if and only if there is an absolute constant c such that*

$$\mu(g^{-1}(A)) \leq c\mu(A) \tag{9.1}$$

for all $\mu-$measurable sets $A \subseteq [0, 2\pi]$ and for $1 < p \leq \infty, 1 \leq q \leq \infty$. Moreover, $\|C_g\| = \|g\|^{1/p}$.

In the theorem,

$$\|g\| \equiv \sup_{\mu(A) \neq 0} \left\{ \frac{\mu(g^{-1}(A))}{\mu(A)} \right\} .$$

CHAPTER 10

EXERCISES

1. Let f, g be continuous functions from (X, d) to (Y, ρ) where X and Y are metric spaces. Show that the set
$$M = \{x \in X : f(x) = g(x)\}$$
is closed in X.

2. Define $d : \mathbb{R}^2 \longrightarrow [0, \infty)$ by $d(x, y) = |x - y|^\alpha$.
Show that $(\mathbb{R}, d)$ is a metric space,$0 < \alpha < 1$.

3. Give an example of two metric spaces that are homeomorphic and one is complete but the other not.

4. Give an example showing that boundedness is not invariant under a homeomorphism.

5. Let ρ be the discrete metric in $\mathbb{R}$, that is,
$$\rho(x, y) = \begin{cases} 0, & x = y; \\ 1, & x \neq y. \end{cases}$$
Describe the Cauchy sequence in $(\mathbb{R}, \rho)$. Is $(\mathbb{R}, \rho)$ a complete metric space?

6. Let X be a vector space d a metric on X so that
$$i)\ d(x + a, y + a) = d(x, y), \ \forall x, y, a \in X$$
$$ii)\ d(\alpha x, 0) = |\alpha| d(x, 0), \ \forall x \in X \ and \ \alpha \in \mathbb{R}.$$
Show that the function $x \longmapsto d(x, 0)$ is a norm in X.

7. Show that if X is a normed space, then X is homeomorphic to
$B_r = \{x \in X : \|x\|_X < r\}$.

8. Let X be the set of positive real numbers with the usual operations of addition and the scalar multiplication given by $\alpha.x = x^{\alpha}$. Is $(X, +, .)$ a vector space?

9. Let X be the set of all functions f defined on $[0, 1]$ so that

$$\int_0^1 \left(e^{|f(t)|} - 1\right) dt < \infty.$$

Endow X with the usual operations of addition and scalar multiplication. Is $(X, +, .)$ a vector space?

10. Give two different proofs showing that the norms in a finite dimensional vector space are equivalents, that is, if $(X, +, .)$ is a finite dimensional vector space and $\|.\|_X$, $\|.\|_X^1$ any two norms in X, then there are positive constants α and β such that

$$\alpha\|x\|_X \leq \|x\|_X^1 \leq \beta\|x\|_X$$

11. Let $M \subset X$ be a vector subspace of the Banach space X. Show that M is a Banach space if and only if M is closed in X.

12. Show that for $0 < p < 1$, $\|f\|_p = \left(\int |f(t)|^p d\mu\right)^{1/p} < \infty$ is not a norm in $(L_p, \mathcal{F}, \mu)$, but if we define d by $d(f, g) = \|f - g\|_p^p$, then d is a metric in L_p.

13. Let $B(X, Y)$ be the set of all bounded linear transformations from X to Y, where X and Y are normed vector spaces with $X \neq \{0\}$. Show that if $B(X, Y)$ is complete, then so is Y.

14. Define $\varphi : l_1 \longrightarrow \mathbb{R}$ where l_1 is endowed with the supremum norm *i.e.*

$$\|x\|_\infty = \sup_{n\geq 1} |x_n|, \quad by \ \ \varphi(x) = \sum_{n=1}^{\infty} x_n.$$

Show that φ is a linear functional that is not bounded.

15. Define $T : l_p \longrightarrow l_p, 1 \leq p \leq \infty$ by $T(x) = (x_2, x_3, ..., x_n, ...)$ where $(x_1, x_2, x_3, ..., x_n, ...) \in l_p$. Show that T is a bounded linear transformation.

16. Let $x_0 \in [0, 1]$ and define $T : C[0, 1] \longrightarrow \mathbb{R}$ by $Tf(x) = f(x_0)$. Endow $C[0, 1]$ with

the supremum norm *i.e.* $\|x\|_\infty = \sup_{0\le t\le 1} |f(t)|$. Show that T is a bounded linear functional on $C[0,1]$.

17. Verify that $C[0,1]$ with the supremum norm is not an inner product space.

18. Is $\mathbb{R}^n$ with the norm $\|x\|_\infty = \max_{1\le\le i\le n} \{|x_i|\}$ an inner product space? Justify your answer.

19. Let $A : C[0,1] \longrightarrow C[0,1]$ where $C[0,1]$ is endowed with the supremum norm, be defined by
$$Af(s) = \int_0^1 k(s,t)f(t)dt.$$
Show that A is a bounded linear transformation and that
$$\|A\| = \sup_{0\le s\le 1} \int_0^1 |k(s,t)|dt$$

20. Let $s > 0$ and $\mathcal{L}f(s) = \displaystyle\int_0^\infty e^{-st} f(t)dt$, well-known as the Laplace Transform. Show that:

    i) $\mathcal{L} : L_\infty \longrightarrow L_\infty$ is a bounded linear transformation.

    ii) $\mathcal{L} : L_1 \longrightarrow L_\infty$ is also a bounded linear transformation.

21. Show that $\mathbb{R}^n$ endowed with the norm $\|x\|_1 = \displaystyle\sum_{i=1}^n |x_i|$; $x = (x_1, x_2, ..., x_n)$ is not an inner product space.

22. If $1 \le 1 \le p_1 \le p_2 \le \infty$, then show that $L_{p_2}(X, \mathcal{F}, \mu) \hookrightarrow L_{p_1}(X, \mathcal{F}, \mu)$, moreover
$$\|f\|_{p_1} \le [\mu(X)]^{\frac{p_2 - p_1}{p_1 p_2}} \|f\|_{p_2},$$
where $(X, \mathcal{F}, \mu)$ is a finite measure space.

23. Show that if $f \in L_{p_1}(X, \mathcal{F}, \mu)$ and $f \in L_{p_2}(X, \mathcal{F}, \mu)$ then $f \in L_p(X, \mathcal{F}, \mu)$ for all $p_1 \le p \le p_2$ and get a relationship between the norms $\|f\|_p$, $\|f\|_{p_1}$ and $\|f\|_{p_2}$.

24. Let $p, r \in [1, \infty)$ with $p \ge r$. Define the product operator T by $Tf = g.f$. Then $T : L_p \longrightarrow L_r$ is bounded if and only if $g \in L_s$ with $\dfrac{1}{s} = \dfrac{1}{r} - \dfrac{1}{p}$.

25. For $1 < p < \infty$, $L_p = L_p((0,\infty))$, $F(x) = \dfrac{1}{x}\displaystyle\int_0^x f(t)dt$. Show that
$$\|F\|_p = \frac{p}{p-1}\|f\|_p.$$

26. Let $(S,\mathcal{A},\mu)$ be a finite measure space and $M(S,\mathcal{A},\mu)$ the set of measurable functions on S. Define in M ρ by
$$\rho(f,g) = \int_X \frac{|f(t)-g(t)|}{1+|f(t)-g(t)|}d\mu(t).$$
Show that
   - $i)$ ρ is a metric over M
   - $ii)$ M is a complete metric space.
   - $iii)$ A sequence $\{f_n\}_{n\geq 1}$ in M converges to f if and only if f_n converges to f in measure. Note that $f_n \to f$ in measure as $n \to \infty$, if $\forall \epsilon > 0$, $\exists N$ so that $\forall s \in S$ such that $|f_n(s) - f(s)| \geq \epsilon$ has measure zero *i.e.*,
$$\mu\{s \in S : \ |f_n(s) - f(s)| \geq \epsilon\} < \epsilon, \ \ \forall n \geq N.$$

27. Show that any linear transformation $T : X \longrightarrow \mathbb{R}^n$ where X is a normed space is bounded.

28. Use exercise 27. to show that any two norms in a finite dimensional vector space are equivalents.

29. Use the closed graph theorem to show that any two norms in a finite dimensional vector space are equivalents.

30. Let $H^1((D))$ be the Hardy space, which is the set of all analytic functions in $\mathbb{D}$ satisfying
$$\|F\|_{H^1} = \sup_{0<r\leq 1}\left(\int_0^{2\pi}|F(re^{i\theta})|d\theta\right) < \infty.$$
Let $B^1(\mathbb{D})$ the analytic form of De Souza's space, which is the set of all analytic

functions in $\mathbb{D}$ satisfying

$$\|F\|_{B^1} = \int_0^1 \int_0^{2\pi} |F'(re^{i\theta})| d\theta dr < \infty. \quad F' \text{ is the derivative of } F$$

Show that $B^1(\mathbb{D})$ is continuously embedded in $H^1(\mathbb{D})$; that is,

$$B^1(\mathbb{D}) \subseteq H^1(\mathbb{D}) \ \ and \ \ \|F\|_{H^1} \leq \|F\|_{B^1}.$$

31. Define

$$B^p = \{f : [0, 2\pi] \to \mathbb{R} : \ f(t) = \sum_{n=1}^{\infty} c_n b_n(t); \ \sum_{n=1}^{\infty} |c_n| \ < \infty$$

where $\{c_n\}_{n\geq 1}$ is a sequence of numbers, $1 < p < \infty$,

$$b_n(t) = \frac{1}{|I_n|^{1/p}} [\chi_{L_n}(t) - \chi_{R_n}(t)],$$

where $I_n = L_n \cup R_n$ is an interval in $[0, 2\pi]$, L_n and R_n the halves of I_n.

$$\|f\|_{B^p} = \inf \sum_{n=1}^{\infty} |c_n|,$$

where the infimum is taken over all possible representations of f. Show that

i) $\|.\|_{B^p}$ is a norm in B^p

ii) B^p is a Banach space.

32. Let

$$\Lambda = \Lambda(1 - \frac{1}{p}, 1, 1) = \{f : [0, 2\pi] \to \mathbb{R} : \ \|f\|_{\Lambda} < \infty\},$$

where

$$\|f\|_{\Lambda} = \int_0^{2\pi} \int_0^{2\pi} \frac{|f(x) - f(y)|}{|x - y|^{2-\frac{1}{p}}} dxdy, \quad 1 < p < \infty.$$

Show that B^p is continuously contained in $\Lambda(1 - \frac{1}{p}, 1, 1)$, that is, $B^p \hookrightarrow \Lambda(1 - \frac{1}{p}, 1, 1)$ for $1 < p < \infty$. Moreover

$$\|f\|_{\Lambda} \leq M \|f\|_{B^p}, \ \ for\ some\ constant\ M > 0.$$

33. Show that B^p as in exercise 31. is continuously contained in the Lebesgue space L_p, that is,
$B^p \hookrightarrow L_p$ for $1 < p < \infty$ and

$$\|f\|_p \leq C\|f\|_{B^p}, \; for\ some\ constant\ C > 0.$$

Moreover show that $\|b\|_{B^p} = 1$ for

$$b(t) = \frac{1}{|I|^{1/p}}[\chi_L(t) - \chi_R(t)],$$

where $I = L \cup R$ is an interval, L and R the halves of I.

34. Define the spaces

$$Lip\alpha = \{f : [0, 2\pi] \to \mathbb{R},\ continuous : \ |f(x+h) - f(x)| \leq Mh^\alpha\}, \; for\ 0 < \alpha \leq 1,$$

$$\|f\|_{Lip_\alpha} = \sup_{h>0,x} \frac{|f(x+h) - f(x)|}{h^\alpha} + \|f\|_\infty$$

and

$$\Lambda_\alpha = \{f : [0, 2\pi] \to \mathbb{R},\ continuous : \ |f(x+h) + f(x-h) - 2f(x)| \leq Mh^\alpha\}, \; for\ 0 < \alpha \leq 2,$$

$$\|f\|_{\Lambda_\alpha} = \sup_{h>0,x} \frac{|f(x+h) + f(x-h) - 2f(x)|}{h^\alpha} + \|f\|_\infty.$$

Lip_α is called the Lipschitz space and Λ_α is the generalized space. Show that,

i) If $\alpha > 1$, then $Lip_\alpha = \{constants\}$.

ii) $\Lambda_1 \subsetneqq Lip_\alpha \subsetneqq Lip_\beta$ for $\alpha < \beta$.

iii) $Lip_\alpha \subsetneqq \Lambda_\alpha$.

iv) $Lip_\alpha = \Lambda_\alpha$ for $0 < \alpha < 1$. Note: Λ_1 is the well-known Zygmund class and is usually denoted in the literature as Λ_*.

v) $(Lip_\alpha, +, ., \|.\|_{Lip_\alpha})$ and $(\Lambda_\alpha, +, ., \|.\|_{\Lambda_\alpha})$ are Banach spaces.

35. Let $L_1(X, \mathcal{F}, \mu)$ be the Lebesgue space and f a function on X. Define

$$m(f, y) = \mu\{x \in X : \ |f(x)| \geq y\},$$

which is well-known as the distribution function of f. Show that

i)
$$\|f\|_1 = \int_X |f(t)|d\mu(t) = \int_0^\infty m(f,y)dy.$$

Note that $m(f,y)$ is the μ-measure of the set $\{x \in X : \ |f(x)| \geq y\}$.

ii) $tm(f,t) \leq \|f\|_1 \; \forall t > 0$.

36. Let $L_p(X, \mathcal{F}, \mu)$ be the Lebesgue space and f a function on X and $m(f,y)$ the distribution function of f. Show that

i)
$$\|f\|_p = \left(\int_X |f(t)|^p d\mu(t)\right)^{1/p} = p\int_0^\infty t^{p-1}m(f,t)dt, \;\; 1 < p < \infty$$

ii) $tm(f,t)^{1/p} \leq \|f\|_p, 1 < p < \infty$.

37. Let's define the weak L_p-spaces, usually denoted by $L(p,\infty)$, as the set

$$L(p,\infty) = \{f : X \to \mathbb{R} : \ tm(f,t)^{1/p} \leq M, \;\; \forall t > 0\}.$$

A "norm" in $L(p,\infty)$ is defined as follows

$$\|f\|_{L(p,\infty)} = \sup_{t>0} tm(f,t)^{1/p}.$$

Problem 36. 2) implies that $L_p \subsetneqq L(p,\infty)$ $1 \leq p < \infty$. Show that the inclusion is proper, that is, there is an $f \in L(p,\infty)$ so that $f \notin L_p$.

38. Let ReH^1 be a space defined as

$$ReH^1 = \{f : X \to \mathbb{R} : \ f(t) = \sum_{n=1}^{\infty} c_n a_n(t); \;\; \sum_{n=1}^{\infty} |c_n| < \infty\},$$

where a_n's are atoms, that is, for any $I_n \subseteq [0, 2\pi]$ intervals,

1) $|a_n(t)| \leq \dfrac{1}{|I_n|}, \; \forall \; t,$
2) $supp\{a_n\} \subseteq I_n$ and

3) $\int_{I_n} a_n(t)dt = 0, \ \ a_0(t) = \frac{1}{2\pi}, \forall\, t.$

Endow ReH^1 with the norm $\|f\|_{ReH^1} = \inf \sum_{n=0}^{\infty} |c_n|$ where the infimum is taken over all representations of f.

Show that

$i)$ $(ReH^1, \|.\|_{ReH^1})$ is a Banach space.

$ii)$ $B^1 \subseteq ReH^1$ where B^1 is the De Souza's space in example 2.24

$iii)$ $ReH^1 \subseteq L_1$, where L_1 is the Lebesgue space.

$iv)$ If $f \in ReH^1$ and $g \in BMO$, (BMO in Example 2.23), then

$$\left| \int_0^{2\pi} f(t)g(t)dt \right| \leq \|f\|_{ReH^1} \|g\|_{BMO}.$$

$v)$ Knowing that $L_2 \subseteq ReH^1$ and $\|f\|_{ReH^1} \leq M\|f\|_2$. Let $\varphi \in (ReH^1)^*$ (the dual of ReH^1), and define λ by $\lambda(A) = \varphi(\chi_A)$ and $L_2 = L_2([0, 2\pi], \mathcal{F}, \mu)$. Show that λ is absolutely continuous with respect to μ.

39. In problem 31. we define the B^p, also in problem 34. we defined the Lip_α and Λ_α, $0 < \alpha < 1$. Now we C^p the space defined on $[0, 2\pi]$ as follows

$$C^p = \{f : [0, 2\pi] \to \mathbb{R} : \ f(t) = \sum_{n=1}^{\infty} c_n d_n(t); \ \ \sum_{n=1}^{\infty} |c_n| < \infty\},$$

where the c_n's are real numbers and $d_n(t) = \frac{1}{|I_n|^{1/p}} \chi_{I_n}(t)$, where I_n's are intervals in $[0, 2\pi]$. Endow C^p with the norm

$$\|f\|_{C^p} = \inf \sum_{n=1}^{\infty} |c_n|$$

where is the infimum is taken over all possible representations of f. Show that

1) $B^p \subseteq C^p$ and $\|f\|_{C^p} \leq M\|f\|_{B^p}$ for $1 < p < \infty$, where M is absolute constant.

2) If $f \in B^p$ and $g \in Lip_{\frac{1}{p}}$, $1 < p < \infty$, then

$$\left| \int_0^{2\pi} f(t)dg(t) \right| \leq \|f\|_{C^p} \|g\|_{Lip_{\frac{1}{p}}}$$

3) If $f \in C^p$ and $g \in \Lambda_{\frac{1}{p}}$, $1 < p < \infty$, then

$$\left| \int_0^{2\pi} f(t)dg(t) \right| \leq \|f\|_{B^p} \|g\|_{\Lambda_{\frac{1}{p}}}.$$

4) $(C^p)^* \cong Lip_{\frac{1}{p}}$, where $(C^p)^*$ is the dual space of C^p. (Hint: use 2))

5) $(B^p)^* \cong \Lambda_{\frac{1}{p}}$, where $(B^p)^*$ is the dual space of B^p. (Hint: use 3)) Note: $dg(t)$ is to be taken in an appropriated sense.

6) Items in 4) and 5) lead to show that

   i) C^p and B^p are equivalents as Banach spaces, that is, $C^p \cong B^p$ with equivalent norms.

   ii) $Lip_\alpha \cong \Lambda_\alpha$, for $0 < \alpha < 1$.

40. Suppose that f is a function from a complete metric space (M, ρ) into itself such that there is $A \in (0,1)$ for which $\rho(f(x), f(y)) \leq A\rho(x,y), \quad \forall\, x, y \in M$. Then, there is a unique $x_0 \in M$ so that $f(x_0) = x_0$.
Note: this theorem is well-known as the fixed point theorem for metric spaces. This is called the **Contraction mapping Theorem**; f is a contraction.

41. Let $S = [0,1]$. endow S with the usual metric *i.e.*, $d(x,y) = |x - y|$. Show that (S, d) is not a complete metric space. Also endow S with the metric ρ defined by $\rho(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|$. Show that indeed ρ is a metric and (S, ρ) is a complete metric space.

42. Show that there is a unique continuous function on $[0,1]$ that satisfies the nonlinear integral equation

$$f(t) - \left(\int_0^t \frac{f(s)}{2} ds \right)^2 = 1, \quad 0 \leq t \leq 1.$$

Hint: Define $B = \{f \in C[0,1] : 1 \leq f(t) \leq 1 + t, \ \forall\, t \in [0,1]\}$. Show that B is closed, then it is complete. Define

$$T : B \longrightarrow B, \ defined\ by\ Tf(t) = 1 + \left(\int_0^t \frac{f(s)}{2} ds \right)^2, \ \forall\, t \in [0,1].$$

Show that T is a contraction, *i.e.*,

$$\|Tf - Tg\|_\infty \leq M \|f - g\|_\infty, \quad M \in (0,1).$$

43. Show that a function satisfying the condition of the contraction mapping theorem is an absolutely continuous function.

44. Let X be an inner product space. Prove that if $x_n \to x$ and $y_n \to y$, then $\langle x_n, y_n \rangle \to \langle x, y \rangle$. Thus the inner product $\langle , \rangle$ is continuous from $X \times X$ into the scalar field.

45. A sequence $\{x_n\}_{n \geq 1}$ in a Hilbert space H is said to converge weakly to x if

$$\lim_{n \to \infty} \langle x_n, y \rangle = \langle x, y \rangle, \quad \forall\, y \in H.$$

 a. Show that the sequence $\{e_n\}_{n \geq 1}$ in l^2 converges weakly to 0 where $e_n = (0, ..., 0, 1, 0, ...)$.
 b. Show that if $\{x_n\}_{n \geq 1}$ converges to x in $\mathbb{R}^n$, then $\{x_n\}_{n \geq 1}$ converges to x in norm.
 c. Show that if $\{x_n\}_{n \geq 1}$ converges weakly to x and $\|x_n\|_H \to \|x\|_H$, then $\{x_n\}_{n \geq 1}$ converges to x in norm.

46. Let $C^n[a,b]$ be the space consisting of all real valued functions f on $[a,b]$ such that the n^{th} derivative of f exists at each $t \in [a,b]$ and is continuous. Denote $C[a,b]$ the space of continuous functions in $[a,b]$ with the supremum norm, *i.e.*, $\|f\|_\infty = \sup\limits_{a \leq t \leq b} |f(t)|$.

 Show that $\|f\|_{C^n[a,b]} = \sum\limits_{k=0}^{n} \|f^{(k)}\|_\infty$, where $f^{(k)}$ represents the k^{th} derivative of f, is a norm, with this norm, $C^n[a,b]$ is a Banach space. Moreover $C^n[a,b]$ is a Banach space isomorphic to $C[a,b] \times \mathbb{R}^n$.

47. the Sobolev space $W_p^n[a,b]$ consist of all functions f defined on $[a,b]$ such that $f^{(n)}$ exists for almost all $t \in [a,b]$ and is in $L_p[a,b]$. Endow W_p^n with the norm

$$\|f\|_{W_p^n} = \sum_{k=0}^{n} \|f^{(k)}\|_p.$$

. Show indeed that $||\cdot||_{W_p^n}$ is a norm. Moreover W_p^n is isomorphic to $L_p[a,b] \times \mathbb{R}^n$. Also show that for $n = 1, 2, 3, ...$ and $r \geq p \geq 1$,

$$C^n \subseteq W_\infty^n \subseteq W_r^n \subseteq W_p^n.$$

48. Show that if we endow W_p^n with

$$\|f\|_* = \sum_{k=0}^{n-1} |f^{(k)}(a)| + \|f^{(n)}\|_p,$$

it is a normed space and, $\|.\|_*$ and $\|.\|_{W_p^n}$ are equivalents.

49. Show that if we endow $C^n[a,b]$ with

$$\|f\|^* = \sum_{k=0}^{n-1} |f^{(k)}(a)| + \|f^{(n)}\|_\infty,$$

it is a normed space and, $\|.\|^*$ and $\|.\|_{C^n}$ are equivalents.

50. Suppose $1 < p < \infty$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Fix $g \in W_q^n$ and define ψ_g by

$$\psi_g(f) = \sum_{j=0}^{n-1} f^{(j)}(a) g^{(j)} + \int_a^b f^{(n)}(t) g^{(n)}(t) dt.$$

Then ψ_g is a bounded linear functional on W_p^n. Moreover every bounded linear functional on W_p^n can be so obtained *i.e.*, the dual space of W_p^n denoted by $(W_p^n)^*$ is W_q^n, *i.e.*, $(W_p^n)^* \cong W_p^n$

51. Let $g \in BVN[a,b]$(normalized bounded variation functions) and define ψ_g by

$$\psi_g(f) = \sum_{j=0}^{n-1} c_j f^{(j)}(a) + \int_a^b f(t)dg(t), \quad \forall\, f \in C^n[a,b]$$

Show that ψ_g is a bounded linear functional on $C^n[a,b]$ and every bounded linear functional on W_p^n can be described in this way *i.e.*, $(C^n[a,b])^* \cong BVN[a,b]$.

52. For each $n \geq 1$, W_2^n is a Hilbert space with the inner product defined by

$$\langle f, g\rangle = \sum_{j=0}^{n} \int_a^b f^{(j)}(t) g^{(j)}(t)dt.$$

Show that $\langle , \rangle$ is an inner product on W_2^n. The inner product determines the norm $\|.\|_*$ defined by

$$\|f\|_* = \left(\sum_{j=0}^{n} \|f^{(j)}\|_2^2\right)^{1/2}.$$

$\|.\|_*$ is equivalent to the original norm defined in W_2^n.

53. Let $T : E \longrightarrow F$ be a bounded linear operator where E and F are normed spaces.
    a. Show that if $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in E, then $\{Tx_n\}_{n\geq 1}$ is a Cauchy sequence in F.
    b. Prove that if F is isomorphic to E and E is complete, then F is also complete.

54. Let E and F be normed spaces. Define in $E \times F$ the following:

$$\|(x,y)\|_\infty = \max\{\|x\|_E, \|y\|_F\},$$

$$\|(x,y)\|_1 = \|x\|_E + \|y\|_F,$$

$$\|(x,y)\|_2 = \left(\|x\|_E^2 + \|y\|_F^2\right)^{1/2}.$$

    a. Show indeed that these are norms and they are equivalents in $E \times F$.
    b. Show that if E and F are Banach spaces, then so is $E \times F$ with the norm $\|.\|_\infty$.

55. Prove that the operator $T(s) = (s_1, s_2 - s_1, s_3 - s_2, ...)$ is an isomorphism, in fact an isometry from bv onto l_1.

Note: $x = (x_n) \in bv$ if $\sum_{n=1}^{\infty} |x_{n+1} - x_n| < \infty$ with norm $\|x\|_{bv} = |x_1| + \sum_{n=1}^{\infty} |x_{n+1} - x_n|$.

56. Let M_1^p be the set of all measurable functions on $[0, 2\pi]$ such that

$$\|f\|_{M_1^p} = \sup_{x>0} \left(\frac{1}{x^{1/p}} \int_0^x f^*(t) t^{1/p-1} dt \right) < \infty \,,$$

where $p > 1$ and f^* is the decreasing rearrangement of f. Show that M_1^p is a characterization of L^∞, the space of bounded functions with sup norm.

57. Let $s = (s_n)_{n\geq 1}$. Prove that the operator $T(s) = (s_1, s_2 - s_1, s_3 - s_2, \ldots)$ is an isometry from bv to ℓ_1. Here bv and ℓ_1 are sets of all sequences s so that

$$\|s\|_{bv} = |s_1| + \sum_{j=1}^{\infty} |s_{j+1} - s_j| < \infty$$

and

$$\|s\|_{\ell_1} = \sum_{j=1}^{\infty} |s_j| < \infty \,,$$

respectively.

58. Define

$$I^p = \{F : \mathbb{D} \to \mathbb{C}, \text{ analytic}\,, \int_0^1 \int_0^{2\pi} |F'(re^{i\theta})|(1-r)^{1/p-1} d\theta dr < \infty\} \,.$$

and

$$b^p = \left\{ F : \mathbb{D} \to \mathbb{D}, F(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \sum_{n=0}^{\infty} 2^n K(n,p) \left(\sum_{k \in I_n} |a_k|^2 \right)^{1/2} < \infty \right\} \,,$$

where $I_n = \{k \in \mathbb{N} : 2^{n-1} \leq k < 2^n\}$ and $K(n, p)$ is a constant. Endow I^p and b^p with the norms

$$\|F\|_{I^p} = \frac{1}{2\pi} \int_0^1 \int_0^{2\pi} |F'(re^{i\theta})|(1-r)^{1/p-1} d\theta dr$$

and

$$\|F\|_{b^p} = \sum_{n=0}^{\infty} 2^n K(n,p) \left(\sum_{k \in I_n} |a_k|^2 \right)^{1/2}.$$

Show that

(a) if $F \in I^p$, then there is a constant $C > 0$ such that $\|F\|_{B^p} \leq C\|F\|_{b^p}$.

(b) If $F \in I^p$ is a lacunary function, then there is a constant $c > 0$ such that $\|F\|_{B^p} \geq c\|F\|_{b^p}$.

Note: Lacunary functions are analytic functions $F(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$ with $\lambda = \inf_k \frac{n_{k+1}}{n_k} > 1$.

59. Let M_r^p be the set of all measurable functions f on $[0, 2\pi]$ such that

$$\|f\|_{M_r^p} = \sup_{x>0} \left\{ \frac{1}{px^{1/p}} \int_0^x \left[f^*(t) t^{1/p} \right]^r \frac{dt}{t} \right\}^{1/r} < \infty$$

for $p > 1$ and $r \geq 1$. Show that M_r^p is the weak $L^{pr'}$ space; that is, $M_r^p \cong L(pr', \infty)$, where $1/r + 1/r' = 1$.

60. If we define

$$M_\phi = \left\{ g : [0, 2\pi] \to \mathbb{R}; \|g\|_{M_\phi} = \sup_{x>0} \int_0^x g^*(t)\phi(t) \frac{dt}{t} < \infty \right\},$$

then $(M_\phi, \|\cdot\|_{M_\phi})$ is a quasi-Banach space. Moreover, $M_\phi \cong L_\infty$ provided that $\phi : [0, 2\pi] \to [0, 2\pi]$ satisfy

(a) ϕ is increasing,

(b) $\frac{\phi(t)}{t}$ is decreasing, and

(c) $\int_0^x \frac{\phi(t)}{t} dt \leq C\phi(x)$.