Some Open Problems in Functional Analysis

Define:

$$f \in B^p \;\Leftrightarrow\; f(t) = \sum c_n \chi_{I_n}(t)\,; \quad \sum |c_n||I_n|^{1/p} < \infty$$

$$f \in B^p \;\Leftrightarrow\; f(t) = \sum c_n \left[\chi_{L_n}(t) - \chi_{R_n}(t)\right]\,; \quad \sum |c_n||I_n|^{1/p} < \infty\,; \quad I_n = L_n \cup R_n$$

$$f \in \Lambda\left(1 - \frac{1}{p}, 1, 1\right) \;\Leftrightarrow\; \|f\|_\Lambda = \int_0^{2\pi}\int_0^{2\pi} \frac{|f(x) - f(y)|}{|x-y|^{2-\frac{1}{p}}} dxdy < \infty$$

$$f \in L(p,1) \;\Leftrightarrow\; \|f\|_{L(p,1)} = \int_0^{2\pi} f^*(t) t^{\frac{1}{p}-1} dt < \infty\,; \quad p > 1$$

$$f \in A(\mu, 1/p) \;\Leftrightarrow\; f(t) = \sum c_n \chi_{A_n}(t)\,; \quad \sum |c_n| \mu(A_n)^{1/p} < \infty\,; \quad p > 1$$

Here $\|g\|_A = \inf \sum |c_n|\mu(A_n)^{1/p}$. Note that $\chi_{A_n}(t)$ can be replaced by $\chi_{B_n}(t) - \chi_{C_n}(t)$ with $C_n \cap B_n = \emptyset$, $A_n = B_n \cup C_n$.

Theorem 1. *$B^p \cong \Lambda\left(1 - \frac{1}{p}, 1, 1\right)$ with equivalent norms.*

Theorem 2 (Multiplication operator on $\Lambda(1 - 1/p, 1, 1)$)**.** *The multiplication operator $T : \Lambda(1 - 1/p, 1, 1) \to \Lambda(1 - 1/p, 1, 1)$, $p > 1$, is bounded if and only if g is bounded almost everywhere and $\forall h \in (0, \pi)$, $\forall a \in [-\pi, \pi]$*

$$\frac{1}{h^{1/p}} \int_0^h \int_a^{a+h} \frac{|g(x+t) - g(x)|}{t^{2-1/p}} dxdt \leq A < \infty\,.$$

Theorem 3 (De Souza, 2010)**.** *$f \in L(p,1) \;\Leftrightarrow\; f \in A(\mu, 1/p)$. Moreover, $\|f\|_{L(p,1)} \cong \|f\|_A$.*

Definition 1. *The Lorentz-Bochner space, denoted by $L^X(p,1)$, where X is a Banach space and $p > 1$*

$$f \in L^X(p,1) \;\Leftrightarrow\; \|f\|_{L^X(p,1)} = \int_0^{2\pi} \|f(t)\|_X^* t^{\frac{1}{p}-1} dt < \infty\,.$$

Open Problems

Define $g : [0, 2\pi] \to [0, 2\pi]$, the composition operator $C_g f = f \circ g$, and the multiplication operator $T_g f = g \cdot f$.

1. Characterize all g so that $C_g : \Lambda\left(1 - \frac{1}{p}, 1, 1\right) \to \Lambda\left(1 - \frac{1}{p}, 1, 1\right)$ boundedly, i.e. $\|C_g f\|_\Lambda \leq M \|f\|_\Lambda$.

2. Characterize all g so that $C_g : \Lambda\left(1 - \frac{1}{p}, 1, 1\right) \to L(p, 1)$ boundedly.

3. Find an atomic decomposition for $L^X(p, 1)$ similarly to De Souza's Theorem.

4. From the atomic decomposition in (3) above, characterize all g so that $C_g f$ is bounded from $L^X(p, 1)$ to $L^X(p, 1)$. Show the same for $T_g f$.