A Finite Element Model for Cell Movement and Deformation

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Our model

- The present work is based on *U. George et al.* and we try to extend their model.
- Reaction-diffusion equations have been used in the modelling of a vast number of phenomena that arise in many areas of the natural sciences such as in geology, ecology, chemistry, physics and Biology
- Here we use it to model the F-actin and G-actin biochemical dynamics on a continuously deforming cell domain.
- The force balance mechanical equation is derived from continuum mechanics and the reaction-diffusion equation is derived from conservation equation exploiting conservation laws .

System of Model Equations

$$
\begin{aligned}
\frac{\partial f}{\partial t} + \bigtriangledown \cdot (f\beta) - D_f \bigtriangleup f &= k_1 a_1 - k_2 f + k_3 f^2 g \\
\frac{\partial g}{\partial t} + \bigtriangledown \cdot (g\beta) - D_g \bigtriangleup g &= k_4 b_1 - k_3 f^2 g \\
\bigtriangledown \cdot (\sigma_v + \sigma_e + \sigma_c + \sigma_p) &= 0
\end{aligned}
$$

where:

$$
\begin{aligned}
\sigma_v &= \mu_1 \frac{\partial \epsilon}{\partial t} + \mu_2 \frac{\partial \phi}{\partial t} I \\
\sigma_e &= \frac{E}{1+v}\left[\epsilon + \frac{v}{1+2v}\phi I\right] \\
\sigma_c &= \psi f^2 exp\left(-\frac{f}{f_{sat}}\right) I = H(f) I \\
\sigma_p &= \frac{p}{1+\phi}\left[1 + \frac{2}{\pi}\delta(\ell) arctan(f)\right] I = \frac{p(f) I}{1+\phi}
\end{aligned}
$$

Non-dimensional of the model

where $\epsilon = \frac{1}{2}(\bigtriangledown \mathbf{u} + \bigtriangledown \mathbf{u}^{\mathbf{T}})$ and $\phi = \bigtriangledown \cdot \mathbf{u}$

$$\overline{f} = \sqrt{\frac{k_3}{k_2}} f, \qquad \overline{g} = \sqrt{\frac{k_3}{k_2}} g, \qquad \overline{f}_{sat} = \sqrt{\frac{k_3}{k_2}} f_{sat},$$

$$\overline{t} = \frac{D_f}{L_x^2} t, \qquad \overline{u} = \frac{u}{L_x}, \qquad \overline{\psi} = \frac{k_2}{k_3} \frac{1+v}{E} \psi,$$

$$\overline{\beta} = \frac{L_x}{D_f} \beta, \qquad \overline{p} = \frac{1+v}{E} p, \qquad a = \sqrt{\frac{k_3}{k_2}} \frac{k_1 a_1}{k_2},$$

$$\overline{\epsilon} = \epsilon, \qquad \overline{\phi} = \phi, \qquad b = \frac{k_4 b_1}{k_2} \sqrt{\frac{k_3}{k_2}},$$

$$\overline{\mu}_i = \mu_i \frac{D_f}{L_x^2} \frac{1+v}{E}, \qquad \overline{\bigtriangledown} = L_x \bigtriangledown, \qquad \overline{\triangle} = L_x^2 \triangle,$$

$$\overline{D}_g = d = \frac{D_g}{D_f}, \qquad \overline{D}_f = 1, \qquad \gamma = \frac{k_2 L_x^2}{D_f}.$$

Non-dimensional form of force balance equation

$$\frac{E}{1+v}\bigtriangledown\cdot\Bigg[\left(\overline{\mu}_1\frac{\partial\overline{\epsilon}}{\partial\overline{t}}+\overline{\mu}_2\frac{\partial\overline{\phi}}{\partial\overline{t}}I\right)+\left(\overline{\epsilon}+\frac{v}{1+2v}\overline{\phi}I\right)$$

$$+\left[\overline{\psi}\;\overline{f}^2\exp\left(-\frac{\overline{f}}{\overline{\overline{f}}_{sat}}\right)I\right]+\frac{v}{1+2v}\overline{\phi}I$$

$$+\frac{\overline{p}}{1+\overline{\phi}}\left[1+\frac{2}{\pi}\delta(\ell)arctan(f)I\right]\Bigg]\;=\;0.$$

$$\Rightarrow\bigtriangledown\cdot\Bigg[\left(\overline{\mu}_1\frac{\partial\overline{\epsilon}}{\partial\overline{t}}+\overline{\mu}_2\frac{\partial\overline{\phi}}{\partial\overline{t}}I\right)+\left(\overline{\epsilon}+\frac{v}{1+2v}\overline{\phi}I\right)+\left[\overline{\psi}\;\overline{f}^2\exp\left(-\frac{\overline{f}}{\overline{\overline{f}}_{sat}}\right)I\right]$$

$$+\frac{v}{1+2v}\overline{\phi}I+\frac{\overline{p}}{1+\overline{\phi}}\left[1+\frac{2}{\pi}\delta(\ell)arctan(f)I\right]\Bigg]\;=\;0.$$

Non-dimensional form for Reaction Diffusion Equations (RDEs)

$$\frac{\partial \overline{f}}{\partial \overline{t}} + \overline{\nabla} \cdot (\overline{f}\ \overline{\beta}) - \overline{\triangle}\ \overline{f} = \gamma(a - \overline{f} + \overline{f}^2 \overline{g})$$

$$\frac{\partial \overline{g}}{\partial \overline{t}} + \overline{\nabla} \cdot (\overline{g}\ \overline{\beta}) - d\overline{\triangle}\ \overline{g} = \gamma(b - \overline{f}^2 \overline{g})$$

Linear stability analysis

The steady state solution of the above system is $f_s = a + b$, $g_s = \frac{b}{(a+b)^2}$ and $\mathbf{u} = \mathbf{0}$. We linearize the system of equations by considering the stability of the steady state to small perturbations $f = f_s + \hat{f}, \quad g = g_s + \hat{g}$, and $\mathbf{u} = \mathbf{u_s} + \hat{\mathbf{u}}$,where $\hat{f}, \hat{g}$ and $\hat{\mathbf{u}}$ are small variations from the steady state.

Substituting these into the nonlinear system above and neglecting all but the linear terms results in the following linear system of partial differential equations:

Linear form of force balance equation

In the next equations, we will drop the hats for the sake of notational convenience

$$\bigtriangledown \cdot [(\mu_1 \epsilon_t + \mu_2 \phi_t I) + (\epsilon + v' \phi I) + H'(f_s) f I - p \phi I + ABfI - A\phi arctan(f_s) I] = 0$$

where $v' = \frac{v}{1-2v}$, $\quad A = \frac{2p}{\pi}\delta(l)$ and $\quad B = \frac{1}{1+f_s^2}$.

Linear form of RDEs

$$f_t = \triangle f - f_s \nabla \cdot (\mathbf{u}_t) + \gamma F_f f + \gamma F_g g$$
$$g_t = d \triangle g - g_s \nabla \cdot (\mathbf{u}_t) + \gamma G_f f + \gamma G_g g$$

where $F_f = 2f_s g_s - 1$, $F_g = f_s^2$, $G_f = 2f_s g_s$, and $G_g = f_s^2$

We now look for solutions to these linearized equations in the form of:

$$f(x,t) = f^* \exp[\lambda t + i\mathbf{k} \cdot \mathbf{x}]$$

$$g(x,t) = g^* \exp[\lambda t + i\mathbf{k} \cdot \mathbf{x}]$$

and

$$\mathbf{u}(x,t) = \mathbf{u}^* \exp[\lambda t + i\mathbf{k} \cdot \mathbf{x}]$$

where λ and $\mathbf{k}$ are respectively the growth rate (also known as an eigenvalue) and the wave vector. Substituting these solutions into the linearized system gives the following linear system of algebraic equations (in vector form):

$$A \cdot \mathbf{x} = \mathbf{0}$$

where A is the matrix

$$\begin{pmatrix} \omega k_1^2 - \frac{\lambda\mu_1 k_2^2 + k^2}{2} & \omega k_1 k_2 & z k_1 & 0 \\ \omega k_1 k_2 & \omega k_2^2 - \frac{\lambda\mu_2 k_1^2 + k^2}{2} & z k_2 & 0 \\ i\lambda f_s k_1 & i\lambda f_s k_2 & \lambda + k^2 - \gamma F_f & -\gamma F_g \\ i\lambda g_s k_1 & i\lambda g_2 k_2 & \gamma G_f & dK^2 + \gamma G_g \end{pmatrix}$$

$$\begin{aligned} \omega &= -\lambda\mu - v'\lambda + p + A\arctan(f_s) + 1/2 \\ z &= i(\psi H'(f_s) + AB) \\ \mu &= \mu_1 + \mu_2 \\ k^2 &= k_1^2 + k_2^2 \end{aligned}$$

and $\mathbf{x} = [u_1^* \quad u_2^* \quad f^* \quad g^*]$

Future Work

- Stability analysis of the linear model
- See if we can identify parameter values necessary for pattern formation
- Use finite element method to compute approximate solutions
- Simulations in Mathematical softwares
- See if the model equations could give rise to cell movement and deformation.

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