Analysis of ODE II

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Outline

1. Modeling Background
2. Phase Plane Analysis
3. The General Setting
4. Advanced Models

Balance Law of Population Dynamics

Balance law

The rate of change of population density equals reproduction rate plus migration rate minus dispersal rate

- Reproduction rate - birth rate minus death rate
- Migration rate - immigration rate minus emigration rate
- Dispersal rate - diffusion (random walk), convective transport, chemotaxis

Models accounting for dispersal lead to partial differential or integro-differential equations

Scope of Talk

Competition of two species

- Closed system, i.e., no migration
- Few comments on dispersal
- Weak law of determinacy, i.e., no memory effects

Geometric approach

Malthus Law

$$\frac{du}{dt} = r\ u(t)$$

Solution

$$u(t) = u(0)e^{rt}$$

- Malthus law - here $r \in \mathbb{R}$ reproduction rate (birth rate minus death rate)
- Examples - bacteria in a petri dish

Model Prediction: prediction of long-term behavior is **unrealistic** if $0 < r$

Logistic Growth

Limited Resources

Replace r by $r(1 - \frac{u}{C})$

- $0 < C$ carrying capacity
- $0 < r$ intrinsic reproduction factor, i.e., reproduction factor in case of unlimited resources.

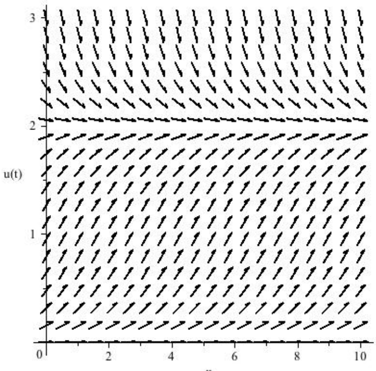
Logistic equation

$$\frac{du}{dt} = ru(t)\left(1 - \frac{u(t)}{C}\right)$$

Logistic Growth

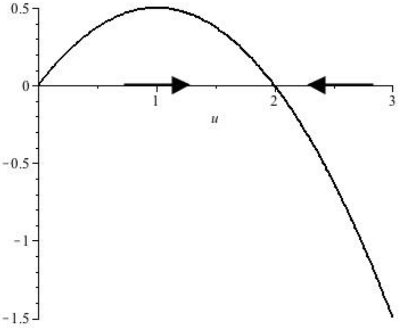
Geometric interpretation

- $F(u) := ru\left(1 - \frac{u}{C}\right)$ is the slope field
- A solution curve $t \mapsto (t, u(t))$ has the slope $(1, F(u(\hat{t}))$ at $(\hat{t}, u(\hat{t}))$



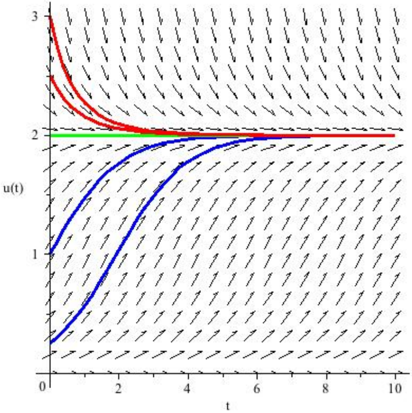
Phase Portrait

Phase diagram



Solution Curves

Solution curves



Two-Species Competition

Two species *densities* u and v

Reproduction rates

- Species 1 $r\left(1 - \frac{u}{K} - \alpha v\right)$
- Species 2 $\rho\left(1 - \frac{v}{L} - \beta u\right)$

Now we have a system of o.d.e.

$$\frac{du}{dt} = r\left(1 - \frac{u}{K} - \alpha v\right) u$$

$$\frac{dv}{dt} = \rho\left(1 - \frac{v}{L} - \beta u\right) v$$

Geometric Interpretation

$$\frac{du}{dt} = u(t)(a_1 - b_1 u(t) - c_1 v(t))$$

$$\frac{dv}{dt} = v(t)(a_2 - b_2 u(t) - c_2 v(t))$$

Solution Curve

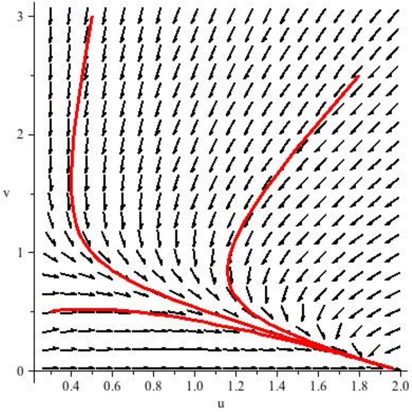
$$t \mapsto \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$$

is a curve in the u, v-plane which is tangent to the slope field

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} u(a_1 - b_1 u - c_1 v) \\ v(a_2 - b_2 u - c_2 v) \end{pmatrix}$$

Example

Slope field and solution curves



Example

Biological Issue co-existence or extinction of one species

Mathematical analysis provides the answer — equilibria and their stability tell the story

Example
Equilibria and Nullclines

Slope Field

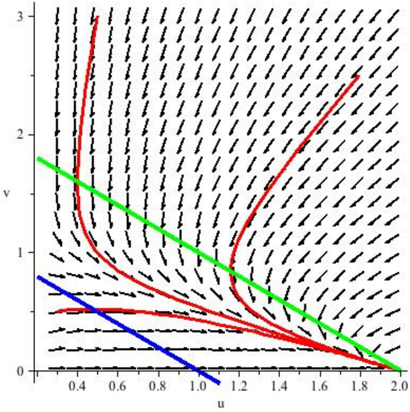
$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} uf(u,v) \\ vg(u,v) \end{pmatrix}$$

Equilibria are the zeroes of the slope field, i.e., points (y, z) such that $yf(y, z) = 0$ and $zg(y, z) = 0$

Nullclines are the sets of all points in phase space (here the first quadrant) where the slope field has horizontal or vertical slope, i.e., either $xf(x, y) = 0$ or $yg(x, y) = 0$

Previous Example Revisited

Slope field, nullclines, and soluteion curves



Equilibria

Solve

$$\begin{aligned} u(a_1 - b_1 u - c_1 v) &= 0 \\ v(a_2 - b_2 u - c_2 v) &= 0 \end{aligned}$$

Solutions $u_0 = 0,\ v_0 = 0$; $u_1 = 0, v_1 = \frac{a_2}{c_2}$; $u_2 = \frac{a_1}{b_1},\ v_2 = 0$; and $u_3 = \frac{a_1 c_2 - a_2 c_1}{b_1 c_2 - b_2 c_1},\ v_3 = -\frac{a_1 b_2 - a_2 b_1}{b_1 c_2 - b_2 c_1}$

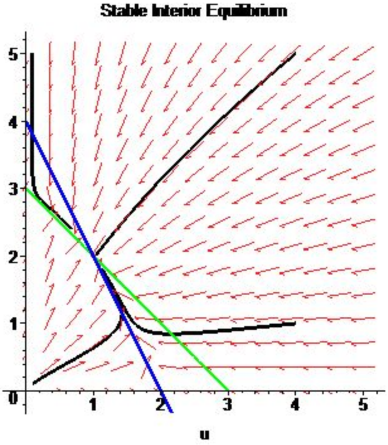
Note (u_3, v_3) belongs to the open first quadrant, if and only if either

$$\frac{b_1}{b_2} > \frac{a_1}{a_2} > \frac{c_1}{c_2}$$

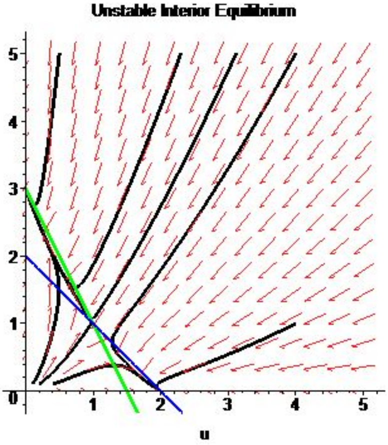
or

$$\frac{b_1}{b_2} < \frac{a_1}{a_2} < \frac{c_1}{c_2}$$

Coexistence of Both Species



Extinction of One Species



Two-Species Competition, General Setting

Consider

$$\begin{cases} \frac{du}{dt} = uf(u,v) \\ \frac{dv}{dt} = vg(u,v) \end{cases}$$

Assumptions

(H0) f, g continuously differentiable functions on $\mathbb{R}^2$

(H1) $f(0,0) > 0$, $g(0,0) > 0$

(H2) $\partial_u f(u,v) < 0$, $\partial_v f(u,v) < 0$, $\partial_u g(u,v) < 0$, $\partial_v g(u,v) < 0$ for $u,\ v \in [0,\infty)$

(H3) there exists a $\beta > 0$ with $f(u,v) < 0$ and $g(u,v) < 0$ for $(u,v) \notin [0,\beta]^2$

Lyapunov Stability

Consider

$$\dot{U}(t) = F(U(t))$$

Hypotheses - $\Omega \subseteq \mathbb{R}^n$ open, $F : \Omega \to \mathbb{R}^n$ continuously differentiable

$U(t; X)$ denotes the unique solution satisfying $U(0) = X$.

Definition - Let $Z \in \Omega$ with $F(Z) = 0$, i.e., Z is an equilibrium of of the eq.

- Z is called *stable*, iff for each $\epsilon > 0$ there exists a $\delta > 0$ such that $|U(t; X) - Z| < \epsilon$ for every $X \in \Omega$ with $|X - Z| < \delta$ and all $t \geq 0$
- Z is called *asymptotically stable*, iff Z is stable and there exists a $\delta > 0$ with $|U(t; X) - Z| \to 0$ as $t \to \infty$, whenever $X \in \Omega$ with $|X - Z| < \delta$
- Z is called UNSTABLE, iff Z is not stable.

Principle of Linearized Stability

Theorem We have

- Z is asymptotically stable provided that the real parts of all eigenvalues of the Jacobian matrix $J_F(Z)$ are negative
- Z is unstable if $J_F(Z)$ has an eigenvalue with positive real part

$$J_F(Z) := \begin{pmatrix} \partial_1 f_1(Z) & \partial_2 f_1(Z) & \dots & \partial_n f_1(Z) \\ \partial_1 f_2(Z) & \partial_2 f_2(Z) & \dots & \partial_n f_2(Z) \\ \vdots & \vdots & \ddots & \\ \partial_1 f_n(Z) & \partial_2 f_n(Z) & \dots & \partial_n f_n(Z) \end{pmatrix}$$

Stability Analysis

Consider

$$\frac{du}{dt} = uf(u, v)$$
$$\frac{dv}{dt} = vg(u, v)$$

Let $(\bar{u}, \bar{v})$ be an equilibrium with $\bar{u}, \bar{v} > 0$, set $F(U) := \big(uf(u, v), vg(u, v)\big)$ for $U = (u, v)$, then

$$J_F((\bar{u}, \bar{v})) = \begin{pmatrix} \bar{u}\partial_1 f(\bar{u}, \bar{v}) & \bar{u}\partial_2 f(\bar{u}, \bar{v}) \\ \bar{v}\partial_1 g(\bar{u}, \bar{v}) & \bar{v}\partial_2 g(\bar{u}, \bar{v}) \end{pmatrix}$$

Eigenvalues

$$\frac{\operatorname{trace}(J_F((\bar{u}, \bar{v}))}{2} \pm \frac{1}{2}\sqrt{\operatorname{trace}(J_F)((\bar{u}, \bar{v}))^2 - 4\det((J_F)((\bar{u}, \bar{v}))}$$

Stability Analysis, Conclusions

Theorem. Let (H1)–(H3) be fulfilled and $(\bar{u}, \bar{v})$ be an isolated interior equilibrium, then $\partial_1 f(\bar{u}, \bar{v}) \partial_2 g(\bar{u}, \bar{v}) > \partial_2 f(\bar{u}, \bar{v}) \partial_1 g(\bar{u}, \bar{v})$ implies asymptotic stability of $(\bar{u}, \bar{v})$, whereas $\partial_1 f(\bar{u}, \bar{v}) \partial_2 g(\bar{u}, \bar{v}) < \partial_2 f(\bar{u}, \bar{v}) \partial_1 g(\bar{u}, \bar{v})$ yields that $(\bar{u}, \bar{v})$ is unstable.

Weak competition allows co-existence

$$\frac{du}{dt} = u(t)(a_1 - b_1 u(t) - c_1 v(t))$$

$$\frac{dv}{dt} = v(t)(a_2 - b_2 u(t) - c_2 v(t))$$

$\frac{b_1}{b_2} > \frac{a_1}{a_2} > \frac{c_1}{c_2}$ co-existence of both species

$\frac{b_1}{b_2} < \frac{a_1}{a_2} < \frac{c_1}{c_2}$ extinction of one species

Modeling Dispersal

Habitat Ω, a bounded open subset of $\mathbb{R}^3$ with smooth boundary

$$\frac{\partial u}{\partial t} - k_1 \Delta u = u f(t, x, u, v)$$

$$\frac{\partial v}{\partial t} - k_2 \Delta v = v g(t, x, u, v)$$

$$\Delta u := \sum_{j=1}^{3} \frac{\partial^2 u}{\partial x_j^2}$$

Boundary Conditions.

- Dirichlet $u, v \equiv 0$ on $\partial\Omega$, lethal boundary
- Neumann $\frac{\partial u}{\partial \mathfrak{n}}, \frac{\partial v}{\partial \mathfrak{n}} \equiv 0$ on $\partial\Omega$, isolated habitat
- Robin $\frac{\partial u}{\partial \mathfrak{n}} + ru \equiv 0$, $\frac{\partial v}{\partial \mathfrak{n}} + \rho v \equiv 0$ on $\partial\Omega$, immigration, emigration

A Justification of the ODE Model

Neumann boundary condition, constant coefficients

$$\frac{\partial u}{\partial t} - k_1 \Delta u = u(a_1 - b_1 u - c_1 v)$$

$$\frac{\partial v}{\partial t} - k_2 \Delta v = v(a_2 - b_2 u - c_2 v)$$

Then (u_3, v_3) is a nontrivial equilibrium, if

$u_3 = \frac{a_1 c_2 - a_2 c_1}{b_1 c_2 - b_2 c_1}, \;\; v_3 = -\frac{a_1 b_2 - a_2 b_1}{b_1 c_2 - b_2 c_1}.$

We have

$\frac{b_1}{b_2} > \frac{a_1}{a_2} > \frac{c_1}{c_2}$ co-existence of both species
$\frac{b_1}{b_2} < \frac{a_1}{a_2} < \frac{c_1}{c_2}$ extinction of one species

Exercise - Picard iteration

Consider the linear i.v.p.

$$u' = 2t(1+u) \qquad u(0) = 0$$

Picard iteration for this equation is

$$u_{k+1}(t) = \int_0^t 2s(1+u_k(s))ds$$

Take $u_0 = 0$ and compute several iterates. *Show* that you get the Taylor series expansion of

$$e^{t^2} - 1$$

which is the exact solution of this i.v.p.