Analysis of ODE I

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Outline

On Mathematical Modeling

Scientific Process

From observation to prediction

- Phenomena and observations
- Basic principles (Physics, Biology, Chemistry, Economy, ...)
- Scientific formulation of the problem
- Mathematical description
- Mathematical analysis
- Computational simulation
- Validation
- Prediction

On Mathematical Modeling

Mathematical models

Applied mathematics and mathematical modeling are devoted to the study and development of methods for prediction of physical systems.

On Mathematical Modeling

Physical Systems

- Physical systems (mechanical, electrical, or electromechanical systems)
- Biological systems
- Financial systems
- Behavioral and social science

On Mathematical Modeling

Why are Models Useful?

- Mathematical models uncover *implicit hypotheses*
- The language of mathematics enables precise communication
- Resources (knowledge and skills) accumulated over hundreds of years are readily available

The Model World

Elements of the model world

- Factors that are neglected
- Factors that affect the model, but are not predicted by the model (exogenous/independent variables and forcing)
- Factors the model is designed to describe (endogenous/dependent variables)

Modeling

Modeling in 5 easy steps

- Ask the question
- Select a modeling approach
- Derive a mathmatical formulation
- Mathematical analysis (and simulation)
- Answer the question (analyze the solution)

Compartmental Model

Spatial variation leads to partial differential equations

To model with o.d.e. you must assume there is no spatial variation

Compartmental model

- Uniformly distributed quantities (e.g. in chemistry: diffusion much faster than reaction; bacteria in a Petri dish)
- Balance (or conservation) law

Compartmental Model

Example: Population Model

Consider a habitat H with a homogeneous population density $P(t)$, e.g. individuals per square mile, surrounded by a much larger habitat with population density $Q(t)$.

Balance law of population dynamics

Rate of change of the population density is equal to growth rate plus migration rate

Compartmental Model

Example: Population Model

Model hypotheses

- Q exogenous variable, P endogenous variable
- Growth rate in H at time t is proportional to the population density at time t
- Migration rate at time t is proportional to $Q(t) - P(t)$ (immigration if the density outside of H is higher than inside H, emigration if the density outside of is smaller than that inside H)

Compartmental Model

Example: Population Model

Question: How does the population P in H evolve over time?

Mathematical formulation

- Growth rate - $rP(t)$, r a constant growth rate
- Migration rate - $m(Q(t) - P(t))$, $0 \leq m$ constant
- Balance law - $\dfrac{dP}{dt} = rP(t) + m(Q(t) - P(t))$

Compartmental Model

Example: Population Model

Mathematical analysis

Study the initial value problem

$$P'(t) + (m - r)P(t) = mQ(t)$$

$$P(0) = P_0$$

where $0 < P_0$ denotes the initial population density at time $t = 0$

Analysis

Existence and Uniqueness

Theorem (local existence)

Let f(t,u) be continuous function for $a < t < b$ and $c < u < d$, and t_0 in (a, b) and u_0 in (c, d). If f is locally Lipschitz continuous in the second argument, uniformly with respect to the first argument, then there exist α and β with $\alpha < t_0 < \beta$ such that the i.v.p.

$$u' = f(t, u)$$
$$u(t_0) = u_0$$

has a unique solution on (α, β)

Analysis

Existence and Uniqueness

Corollary (global existence)

Under the previous assumptions, if the i.v.p. is linear, i.e., $f(t,u) = a(t)u + g(t)$ then the i.v.p. has a unique solution on $\mathbb{R}$.

Analysis
Picard Iteration

Consider

$$u' = f(t, u)$$
$$u(t_0) = u_0$$

the i.v.p. can be turned into an equivalent integral equation by integrating the d.e.

$$u(t) = u_0 + \int_{t_0}^{t} f(s, u(s))ds$$

Define the Picard iteration (fixed point iteration)

$$u_{k+1}(t) = u_0 + \int_{t_0}^{t} f(s, u_k(s))ds \qquad k = 0, 1, 2, \ldots$$

Back to Example

Compartmental Model

Example: Population Model

The equation $P'(t) + (m - r)P(t) = mQ(t)$ is a linear d.e., hence the solution of the i.v.p.

$$P'(t) + (m - r)P(t) = mQ(t)$$

$$P(0) = P_0$$

is

$$P(t) = e^{(r-m)t}\left(P_0 + m\int_0^t Q(s)e^{(m-r)s}ds\right)$$

Back to Example

Compartmental Model

Example: Population Model

Answer the question

Case study:

Isolated habitat

- Migration constant $m = 0$
- $P(t) = P_0 e^{rt}$
- Growth rate r = birth rate - death rate
  - $0 < r$ exponential growth, unrealistic in view of limited resources
  - $r = 0$ population density is constant
  - $r < 0$ exponential decay

Back to Example

Compartmental Model

Example: Population Model

Recall

$$P(t) = e^{(r-m)t}\left(P_0 + m\int_0^t Q(s)e^{(m-r)s}ds\right)$$

Migration rate equals growth rate

- $m \neq 0$
- $P(t) = P_0 + m\int_0^t Q(s)ds$
- $\int_0^\infty Q(s)ds < \infty$ is necessary for P to remain bounded, which is unrealistic

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Compartmental Model

Example: Population Model

Recall

$$P(t) = e^{(r-m)t} \left(P_0 + m \int_0^t Q(s) e^{(m-r)s} ds \right)$$

Outside population density constant

- $m \neq 0$, $m \neq r$, Q constant

$$\begin{aligned} P(t) &= e^{(r-m)t} \left(P_0 + mQ \int_0^t e^{(m-r)s} ds \right) \\ &= \left(P_0 - \frac{mQ}{m-r} \right) e^{(r-m)t} + \frac{mQ}{m-r} \end{aligned}$$

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Compartmental Model

Example: Population Model

Outside population density constant

- $m \neq 0$, $m \neq r$, Q constant

$$P(t) = \left(P_0 - \frac{mQ}{m-r}\right) e^{(r-m)t} + \frac{mQ}{m-r}$$

- Migration and growth rate $m < r$, $|P(t)| \to \infty$ as $t \to \infty$, which is unrealistic
- Migration and growth rate $r < m$, $P(t) \to \frac{mQ}{m-r}$ as $t \to \infty$

Compartmental Model

Example: Population Model

Conclusion: we need a better model

In fact we need to add that resources are limited

Logistic-type equation?

$$\frac{dP}{dt} = r\left(1 - \frac{P(t)}{C}\right) + m(Q(t) - P(t))$$