Numerical Analysis and Methods for PDE II

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Outline

Weak and Variational Formulations

Define

$$(u, v) = \int_0^1 u(x)v(x)\,dx$$

and

$$F(v) = \frac{1}{2}[(v', v') + (cv, v)] - (\hat{f}, v)$$

Weak and Variational Formulations - Continued

The weak problem is find $\hat{u} \in V$ such that

$$(\hat{u}', v') + (c\hat{u}, v) = (\hat{f}, v) \quad \text{for all} \quad v \in V$$

The variational problem is find $\hat{u} \in V$ such that

$$F(\hat{u}) \leq F(v) \quad \text{for all} \quad v \in V$$

Then $u = \hat{u} + g$ is a weak (or variational) solution of the original problem
If u is sufficiently regular, it is a strong solution, or a classical solution of the p.d.e.

Comments on Existence and Uniqueness

Find $u \in V$ such that

$$(u', v') + (cu, v) = (f, v) \quad \text{for all} \quad v \in V$$

(Note we dropped the hat ˆ notation)

Continuity and Stability of Forms

On V we have an inner product

$$\langle u, v \rangle = (u', v') + (cu, v)$$

or

$$\langle u, v \rangle = (u', v')$$

and a norm

$$\|v\| = \sqrt{(v', v') + (cv, v)}$$

or

$$\|v\| = \sqrt{(v', v')}$$

Continuity and Stability of Forms

Continuity

$$(u', v') + (cu, v) \leq C\|u\|\|v\|$$

and

$$(f, v) \leq M\|v\|$$

Stability

$$(v', v') + (cv, v) \geq \alpha\|v\|^2$$

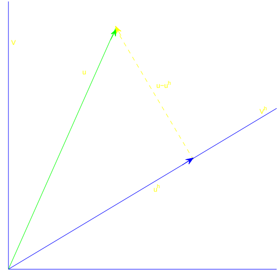
Existence and Uniqueness

We can show the weak problem has a unique solution.
Proof...

Best Approximation

Given V^h the finite element approximation u^h of u satisfies

$$\|u - u^h\|_{H^1} \leq C\|u - v^h - g^h\|_{H^1} \qquad \text{for all } v^h \in V^h$$



Best Approximation - The Proof

$$\begin{aligned}\alpha\|u-u^h\|_{H^1}^2 &\leq ((u-u^h)',(u-u^h)')+(c(u-u^h),u-u^h)\\ &=((u-u^h)',(u-v^h-g^h)')+(c(u-u^h),u-v^h-g^h)\\ &\leq M\|u-u^h\|_{H^1}\|u-v^h-g^h\|_{H^1}\end{aligned}$$

$$\|u-u^h\|_{H^1}\leq C\|u-v^h-g^h\|_{H^1} \qquad \text{for all } v^h\in V^h$$

Extensions

- Time dependent problems (finite differences, or discontinuous Galerkin)
- Nonconforming finite elements
- Saddle point problems (mixed formulations, inf-sup)
- First order systems

2-d Problem

Consider

$$-\Delta u + cu = f \qquad \text{in } \Omega$$

$$u|_{\partial\Omega} = g$$

Homogenize the problem, find $\hat{g}$ such that $\hat{g}|_{\partial\Omega} = g$

Find $\hat{u}$ such that

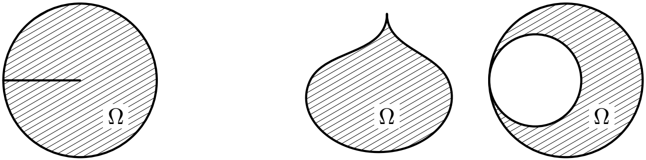
$$-\Delta \hat{u} + c\hat{u} = \hat{f} \qquad \text{in } \Omega$$

$$\hat{u}|_{\partial\Omega} = 0$$

where

$$\hat{f} = f + \Delta\hat{g} - c\hat{g}$$

The Domain



Usually require Ω to be of class $C^{0,1}$ or $C^{1,1}$, satisfy the cone condition, and be on one side of the boundary.

Weak Form

Multiply the p.d.e. by a *test function* v and integrate over Ω

$$\int_\Omega [-\Delta \hat{u} + c\hat{u}] v \, dx = \int_\Omega \hat{f} v \, dx$$

v in some appropriate function space[1] V

Integrating by parts we get

$$\begin{aligned} \int_\Omega (-\Delta \hat{u} + c\hat{u}) v \, dx &= -\int_{\partial\Omega} \nabla \hat{u} \cdot n v \, ds + \int_\Omega \nabla \hat{u} \cdot \nabla v + c\hat{u} v \, dx \\ &= \int_\Omega \nabla \hat{u} \cdot \nabla v + c\hat{u} v \, dx \end{aligned}$$

[1]Here $V = H_0^1(\Omega)$

Weak and Variational Formulations

Set

$$(u, v) = \int_{\Omega} uv\,dx$$

and

$$F(v) = \frac{1}{2}[(\nabla v, \nabla v) + (cv, v)] - (\hat{f}, v)$$

Weak and Variational Formulations - Continued

The weak problem is find $\hat{u} \in V$ such that

$$(\nabla \hat{u}, \nabla v) + (c\hat{u}, v) = (\hat{f}, v) \quad \text{for all} \quad v \in V$$

The variational problem is find $\hat{u} \in V$ such that

$$F(\hat{u}) \leq F(v) \quad \text{for all} \quad v \in V$$

Then $u = \hat{u} + \hat{g}$ is a solution of the original problem

Discrete Problem

Construct a finite dimensional space[1] $V^h \subset V$

Solve the discrete weak (or variational) problem find $\hat{u}^h \in V^h$ such that

$$(\nabla \hat{u}^h, \nabla v^h) + (c\hat{u}^h, v^h) = (\hat{f}, v^h) \quad \text{for all } v^h \in V^h$$

or find $\hat{u}^h \in V$ such that

$$F(\hat{u}^h) \leq F(v^h) \quad \text{for all } v^h \in V^h$$

Then $\hat{u}^h + \hat{g}$ is an approximation to u the solution of the p.d.e.

[1] This leads to, so called, conforming finite element methods, if this inclusion does not hold we have nonconforming finite elements

Finite Element Method

Main tasks

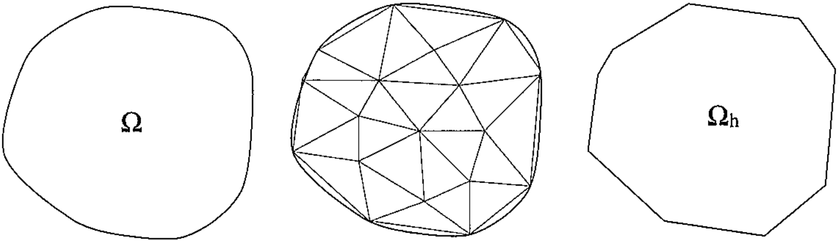
- Construct a basis for V^h
- Construct the discrete weak form
- Solve algebraic system of equations

Finite Element Method

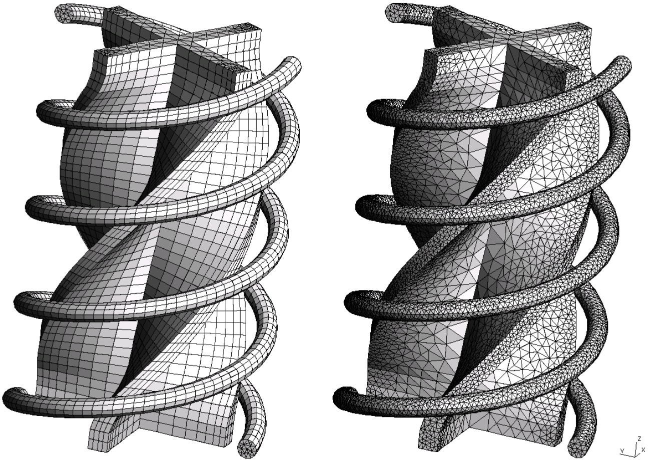
List of Ingredients

- Mesh generator (data structure)
- Basis functions
- Assembly routines
- Algebraic solver (linear solver)
- Post processor

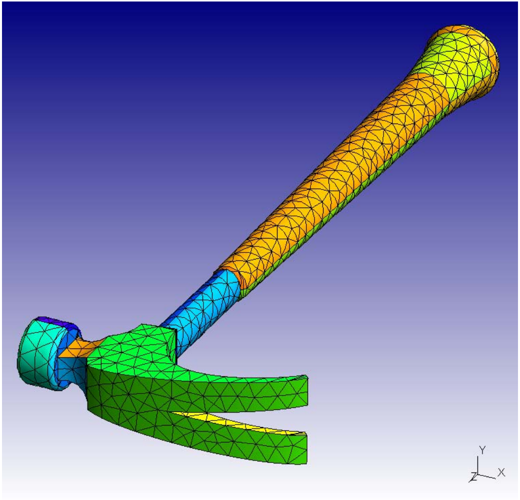
Mesh Generator



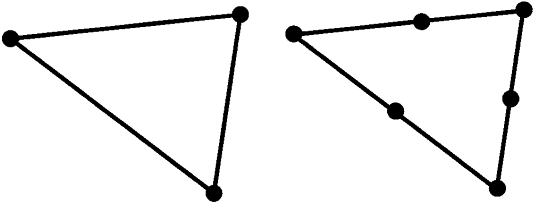
Mesh Generator



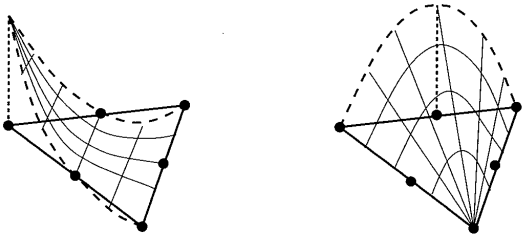
Mesh Generator



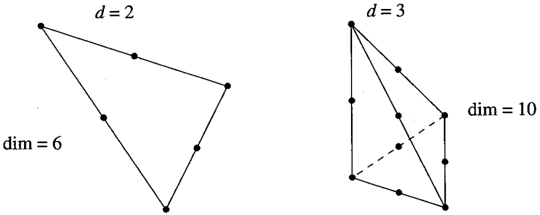
Elements



Basis Functions

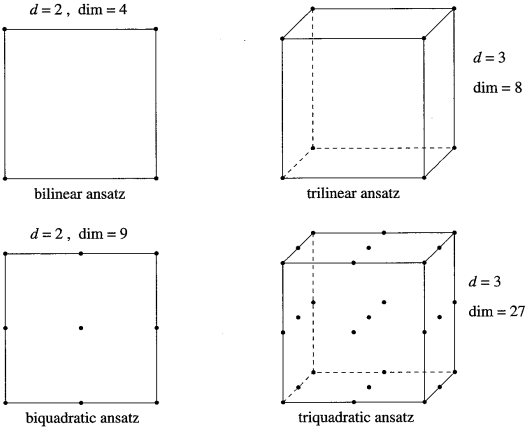


Elements



Quadratic simplicial elements.

Elements



Quadratic and cubic elements on the cube.

Assembly

$$\sum_{i=1}^{n} \hat{u}_i^h [(\phi_i', \phi_j') + (c\phi_i, \phi_j)] = (\hat{f}, \phi_j) \quad \text{for} \quad 1 \leq j \leq n$$

Algebraic Solver

Linear solver (direct, or iterative)
Nonlinear solver (iterative)

Post Processor